

# Yang-Baxter basis of Hecke algebra and Casselman's problem (extended abstract)

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**Abstract.** We generalize the definition of Yang-Baxter basis of type  $A$  Hecke algebra introduced by A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) to all the Lie types and prove their duality. As an application we give a solution to Casselman's problem on Iwahori fixed vectors of principal series representation of  $p$ -adic groups.

**Résumé.** Nous généralisons la définition de la base de l'algèbre de Hecke de Yang-Baxter de type  $A$  introduit par A.Lascoux, B.Leclerc and J.Y.Thibon (Letters in Math. Phys., 40 (1997), 75–90) à tous les types de Lie et prouvons la dualité. Comme application nous donnons une solution du problème de Casselman sur les vecteurs fixés par le sous-groupe d'Iwahori de la série principale des groupes  $p$ -adiques.

**Keywords.** Yang-Baxter basis, Iwahori fixed vector, Casselman's problem, Demazure-Lusztig operator

## 1 Introduction

Yang-Baxter basis of Hecke algebra of type  $A$  was defined in the paper of Lascoux-Leclerc-Thibon [LLT]. There is also a modified version in [Las]. First we generalize the latter version to all the Lie types. Then we will solve the Casselman's problem on the basis of Iwahori fixed vectors using Yang-Baxter basis and Demazure-Lusztig type operator. This paper is an extended abstract and the detailed proofs will appear in [NN].

## 2 Generic Hecke algebra

### 2.1 Root system, Weyl group and generic Hecke algebra

Let  $\mathcal{R} = (\Lambda, \Lambda^*, R, R^*)$  be a (reduced) semisimple root data cf. [Dem]. More precisely  $\Lambda \simeq \mathbb{Z}^r$  is a weight lattice with  $\text{rank } \Lambda = r$ . There is a pairing  $\langle \cdot, \cdot \rangle: \Lambda^* \times \Lambda \rightarrow \mathbb{Z}$ .  $R \subset \Lambda$  is a root system with simple roots  $\{\alpha_i\}_{1 \leq i \leq r}$  and positive roots  $R^+$ .  $R^* \subset \Lambda^*$  is the set of coroots, and there is a bijection  $R \rightarrow R^*$ ,  $\alpha \mapsto \alpha^*$ . We also denote the coroot  $\alpha^* = h_\alpha$ . The Weyl group  $W$  of  $\mathcal{R}$  is generated by

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simple reflections  $S = \{s_i\}_{1 \leq i \leq r}$ . The action of  $W$  on  $\Lambda$  is given by  $s_i(\lambda) = \lambda - \langle \alpha_i^*, \lambda \rangle \alpha_i$  for  $\lambda \in \Lambda$ . We define generic Hecke algebra  $H_{t_1, t_2}(W)$  over  $\mathbb{Z}[t_1, t_2]$  with two parameters  $t_1, t_2$  as follows. Generators are  $h_i = h_{s_i}$ , with relations  $(h_i - t_1)(h_i - t_2) = 0$  for  $1 \leq i \leq r$  and the braid relations  $\underbrace{h_i h_j \cdots}_{m_{i,j}} = \underbrace{h_j h_i \cdots}_{m_{i,j}}$ , where  $m_{i,j}$  is the order of  $s_i s_j$  for  $1 \leq i < j \leq r$ . We need to extend the coefficients

to the quotient field of the group algebra  $\mathbb{Z}[\Lambda]$ . An element of  $\mathbb{Z}[\Lambda]$  is denoted as  $\sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ . The Weyl group acts on  $\mathbb{Z}[\Lambda]$  by  $w(e^\lambda) = e^{w\lambda}$ . We extend the coefficient ring  $\mathbb{Z}[t_1, t_2]$  of  $H_{t_1, t_2}(W)$  to

$$Q_{t_1, t_2}(\Lambda) := \mathbb{Z}[t_1, t_2] \otimes Q(\mathbb{Z}[\Lambda])$$

where  $Q(\mathbb{Z}[\Lambda])$  is the quotient field of  $\mathbb{Z}[\Lambda]$ .

$$H_{t_1, t_2}^{Q(\Lambda)}(W) := Q_{t_1, t_2}(\Lambda) \otimes_{\mathbb{Z}[t_1, t_2]} H_{t_1, t_2}(W).$$

For  $w \in W$ , an expression of  $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$  with minimal number of generators  $s_{i_k} \in S$  is called a reduced expression in which case we write  $\ell(w) = \ell$  and call it the length of  $w$ . Then  $h_w = h_{i_1} h_{i_2} \cdots h_{i_\ell}$  is well defined and  $\{h_w\}_{w \in W}$  forms a  $Q_{t_1, t_2}^{Q(\Lambda)}$ -basis of  $H_{t_1, t_2}^{Q(\Lambda)}(W)$ .

### 2.2 Yang-Baxter basis and its properties

Yang-Baxter basis was introduced in the paper [LLT] to investigate the relation with Schubert calculus. There is also a variant in [Las] for type  $A$  case. We generalize that results to all Lie types.

For  $\lambda \in \Lambda$ , we define  $E(\lambda) = e^{-\lambda} - 1$ . Then  $E(\lambda + \nu) = E(\lambda) + E(\nu) + E(\lambda)E(\nu)$ . In particular, if  $\lambda \neq 0$ ,  $\frac{1}{E(\lambda)} + \frac{1}{E(-\lambda)} = -1$ .

**Proposition 1** For  $\lambda \in \Lambda$ , if  $\lambda \neq 0$ , let  $h_i(\lambda) := h_i + \frac{t_1 + t_2}{E(\lambda)}$ . Then these satisfy the **Yang-Baxter relations**, i.e. if we write  $[p, q] := p\lambda + q\nu$  for fixed  $\lambda, \nu \in \Lambda$ , the following equations hold. We assume all appearance of  $[p, q]$  is nonzero.

$$\begin{aligned} h_i([1, 0])h_j([0, 1]) &= h_j([0, 1])h_i([1, 0]) && \text{if } m_{i,j} = 2 \\ h_i([1, 0])h_j([1, 1])h_i([0, 1]) &= h_j([0, 1])h_i([1, 1])h_j([1, 0]) && \text{if } m_{i,j} = 3 \\ h_i([1, 0])h_j([1, 1])h_i([1, 2])h_j([0, 1]) &= h_j([0, 1])h_i([1, 2])h_j([1, 1])h_i([1, 0]) && \text{if } m_{i,j} = 4 \\ h_i([1, 0])h_j([1, 1])h_i([2, 3]) &= h_j([0, 1])h_i([1, 3])h_j([1, 2]) \\ \times h_j([1, 2])h_i([1, 3])h_j([0, 1]) &= \times h_i([2, 3])h_j([1, 1])h_i([1, 0]) && \text{if } m_{i,j} = 6 \end{aligned}$$

**Proof:** We can prove these equations by direct calculations. □

**Remark 1** In [Che] I. Cherednik treated Yang-Baxter relation in more general setting. There is also a related work [Kat] by S. Kato and the proof of Theorem 2.4 in [Kat] suggests a uniform way to prove Yang-Baxter relations without direct calculations.

We use the Bruhat order  $x \leq y$  on elements  $x, y \in W$  (cf.[Hum]). Following [Las] we define the Yang-Baxter basis  $Y_w$  for  $w \in W$  recursively as follows.

$$Y_e := 1, Y_w := Y_{w'}(h_i + \frac{t_1+t_2}{w'E(\alpha_i)}) \text{ if } w = w's_i > w'.$$

Using the Yang-Baxter relation above it is easy to see that  $Y_w$  does not depend on a reduced expression of  $w$ . As the leading term of  $Y_w$  with respect to the Bruhat order is  $h_w$ , they also form a  $Q_{t_1,t_2}(\Lambda)$ -basis  $\{Y_w\}_{w \in W}$  of  $H_{t_1,t_2}^{Q(\Lambda)}(W)$ . We are interested in the transition coefficients  $p(w, v)$  and  $\tilde{p}(w, v) \in Q_{t_1,t_2}(\Lambda)$  between the two basis  $\{Y_w\}_{w \in W}$  and  $\{h_w\}_{w \in W}$ , i.e.

$$Y_v = \sum_{w \leq v} p(w, v)h_w, \text{ and } h_v = \sum_{w \leq v} \tilde{p}(w, v)Y_w.$$

Take a reduced expression of  $v$  e.g.  $v = s_{i_1} \cdots s_{i_\ell}$  where  $\ell = \ell(v)$  is the length of  $v$  (cf. [Hum]). Then  $Y_v$  is expressed as follows.

$$Y_v = \prod_{j=1}^{\ell} \left( h_{i_j} + \frac{t_1 + t_2}{E(\beta_j)} \right)$$

where  $\beta_j := s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$  for  $j = 1, \dots, \ell$ . The set  $R(v) := \{\beta_1, \dots, \beta_\ell\} \subset R^+$  is independent of the reduced expression of  $v$ . The Yang-Baxter basis defined in [LLT] is normalized as follows.

$$Y_v^{LLT} := \left( \prod_{j=1}^{\ell} \frac{E(\beta_j)}{t_1 + t_2} \right) Y_v = \prod_{j=1}^{\ell} \left( \frac{E(\beta_j)}{t_1 + t_2} h_{i_j} + 1 \right).$$

**Remark 2** *The relation to K-theory Schubert calculus is as follows. If we set  $t_1 = 0, t_2 = -1$  and replacing  $\alpha_i$  by  $-\alpha_i$ . Then the coefficient of  $h_w$  in  $Y_v^{LLT}$  is the localization  $\psi^w(v)$  at  $v$  of the equivariant K-theory Schubert class  $\psi^w$  (cf. [LSS]).*

Let  $w_0$  be the longest element in  $W$ . Define  $Q_{t_1,t_2}(\Lambda)$ -algebra homomorphism  $\Omega : H_{t_1,t_2}^{Q(\Lambda)} \rightarrow H_{t_1,t_2}^{Q(\Lambda)}$  by  $\Omega(h_w) = h_{w_0 w w_0}$ . Let  $\star$  be the ring homomorphism on  $\mathbb{Z}[\Lambda]$  induced by  $\star(e^\lambda) = e^{-\lambda}$  and extend to  $Q_{t_1,t_2}(\Lambda)$ .

**Proposition 2** (Lascoux [Las] Lemma 1.8.1 for type A case) *For  $v \in W$ ,*

$$\Omega(Y_{w_0 v w_0}) = \star[w_0(Y_v)]$$

where  $W$  acts only on the coefficients.

**Proof:** When  $\ell(v) > 0$  there exists  $s \in S$  such that  $v = v's > v'$ . Using the induction assumption on  $v'$ , we get the formula for  $v$ . □

Taking the coefficient of  $h_w$  in the above equation, we get

**Corollary 1**

$$p(w_0 w w_0, w_0 v w_0) = \star[w_0 p(w, v)].$$

### 2.3 Inner product and orthogonality

Define inner product  $(\ , \ )^H$  on  $H_{t_1, t_2}^{Q(\Lambda)}(W)$  by  $(f, g)^H :=$  the coefficient of  $h_{w_0}$  in  $fg^\vee$ , where  $g^\vee = \sum c_w h_{w^{-1}}$  if  $g = \sum c_w h_w$ . It is easy to see that  $(fh_s, g)^H = (f, gh_s)^H$  for  $f, g \in H_{t_1, t_2}^{Q(\Lambda)}(W)$  and  $s \in S$ . There is an involution  $\hat{\cdot} : H_{t_1, t_2}^{Q(\Lambda)} \rightarrow H_{t_1, t_2}^{Q(\Lambda)}$  defined by  $\hat{h}_i = h_i - (t_1 + t_2), \hat{t}_1 = -t_2, \hat{t}_2 = -t_1$ . It is easy to see that  $\hat{h}_s h_s = -t_1 t_2$  for  $s \in S$ .

The following proposition is due to A.Lascoux for the type  $A$  case [Las] P.33.

**Proposition 3** For all  $v, w \in W$ ,

$$(h_v, \hat{h}_{w_0 w})^H = \delta_{v, w}.$$

**Proof:** We can use induction on the length  $\ell(v)$  of  $v$  to prove the equation. □

We have another orthogonality between  $Y_v$  and  $w_0(Y_{w_0 w})$ .

**Proposition 4** (Type  $A$  case was due to [LLT] Theorem 5.1, [Las] Theorem 1.8.4.)

For all  $v, w \in W$ ,

$$(Y_v, w_0(Y_{w_0 w}))^H = \delta_{v, w}.$$

**Proof:** We use induction on  $\ell(v)$  and use the fact that if  $s \in S$  and  $u \in W$ , then  $Y_u h_s = aY_{us} + bY_s$  for some  $a, b \in Q_{t_1, t_2}(\Lambda)$ . □

### 2.4 Duality between the transition coefficients

Recall that we have two transition coefficients  $p(w, v), \tilde{p}(w, v) \in Q_{t_1, t_2}(\Lambda)$  defined by the following expansions.

$$Y_v = \sum_{w \leq v} p(w, v) h_w$$

$$h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$$

Below gives a relation between them.

**Theorem 1** (Lascoux [Las] Corollary 1.8.5 for type  $A$  case) For  $w, v \in W$ ,

$$\tilde{p}(w, v) = (-1)^{\ell(v) - \ell(w)} p(vw_0, ww_0).$$

**Proof:** We will calculate  $(h_v, w_0(Y_{w_0 w}))^H$  in two ways. As  $h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$ ,

$$(h_v, w_0(Y_{w_0 w}))^H = \tilde{p}(w, v)$$

by the orthogonality on  $Y_v$  (Proposition 4). On the other hand, as  $h_i + \frac{t_1 + t_2}{E(\beta)} = \hat{h}_i - \frac{t_1 + t_2}{E(-\beta)}$  for  $\beta \in R$ , we can expand  $Y_v$  in terms of  $\hat{h}_w$  as follows.

$$Y_v = \sum_{w \leq v} (-1)^{\ell(v) - \ell(w)} \star [p(w, v)] \hat{h}_w.$$

So we have

$$w_0(Y_{w_0 w}) = \sum_{w_0 v \leq w_0 w} (-1)^{\ell(v) - \ell(w)} w_0[\star p(w_0 v, w_0 w)] \hat{h}_{w_0 v}.$$

Then using the orthogonality on  $h_v$  (Proposition 3) and Corollary 1,

$$(h_v, w_0(Y_{w_0 w}))^H = (-1)^{\ell(v) - \ell(w)} w_0[\star p(w_0 v, w_0 w)] = (-1)^{\ell(v) - \ell(w)} p(vw_0, ww_0).$$

The theorem is proved. □

### 2.5 Recurrence relations

Here we give some recurrence relations on  $p(w, v)$  and  $\tilde{p}(w, v)$ .

**Proposition 5** (left  $p$ ) For  $w \in W$  and  $s \in S$ , if  $sv > v$  then

$$p(w, sv) = \begin{cases} \frac{t_1+t_2}{E(\alpha_s)} s[p(w, v)] - t_1 t_2 s[p(sw, v)] & \text{if } sw > w \\ (t_1 + t_2) \left( \frac{1}{E(\alpha_s)} + 1 \right) s[p(w, v)] + s[p(sw, v)] & \text{if } sw < w. \end{cases}$$

**Proof:** By the definition we have  $Y_{sv} = Y_s s[Y_v]$  from which we can deduce the recurrence formula. □

We note that by this recurrence we can identify  $p(w, v)$  as a coefficient of transition between two bases of the space of Iwahori fixed vectors cf. Theorem 3 below.

**Proposition 6** (right  $p$ ) For  $w \in W$  and  $s \in S$ , if  $vs > v$  then

$$p(w, vs) = \begin{cases} \frac{t_1+t_2}{vE(\alpha_s)} p(w, v) - t_1 t_2 p(ws, v) & \text{if } ws > w \\ (t_1 + t_2) \left( \frac{1}{vE(\alpha_s)} + 1 \right) p(w, v) + p(ws, v) & \text{if } ws < w. \end{cases}$$

**Proof:** We can use the equation  $Y_{vs} = Y_v v[Y_s]$  and taking the coefficient of  $h_w$ , we get the formula. □

**Proposition 7** (left  $\tilde{p}$ ) For  $w \in W$  and  $s \in S$ , if  $sv > v$  then

$$\tilde{p}(w, sv) = \begin{cases} -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w, v) + \left(1 + \frac{t_1+t_2}{E(\alpha_s)}\right) \left(1 + \frac{t_1+t_2}{E(-\alpha_s)}\right) s[\tilde{p}(sw, v)] & \text{if } sw > w \\ -\frac{t_1+t_2}{E(\alpha_s)} \tilde{p}(w, v) + s[\tilde{p}(sw, v)] & \text{if } sw < w. \end{cases}$$

**Proof:** We can prove the recurrence relation using Corollary 2 below. □

**Proposition 8** (right  $\tilde{p}$ ) For  $w \in W$  and  $s \in S$ , if  $vs > v$  then

$$\tilde{p}(w, vs) = \begin{cases} -\frac{t_1+t_2}{wE(\alpha_s)} \tilde{p}(w, v) + \left(1 + \frac{t_1+t_2}{wE(\alpha_s)}\right) \left(1 + \frac{t_1+t_2}{wE(-\alpha_s)}\right) \tilde{p}(ws, v) & \text{if } ws > w \\ -\frac{t_1+t_2}{wE(\alpha_s)} \tilde{p}(w, v) + \tilde{p}(ws, v) & \text{if } ws < w. \end{cases}$$

**Proof:** We can prove the recurrence relation using Corollary 2 below. □

### 3 Kostant-Kumar's twisted group algebra

Let  $Q_{t_1, t_2}^{KK}(W) := Q_{t_1, t_2}(\Lambda) \# \mathbb{Z}[W]$  be the (generic) twisted group algebra of Kostant-Kumar. Its element is of the form  $\sum_{w \in W} f_w \delta_w$  for  $f_w \in Q_{t_1, t_2}(\Lambda)$  and the product is defined by

$$\left( \sum_{w \in W} f_w \delta_w \right) \left( \sum_{u \in W} g_u \delta_u \right) = \sum_{w, u \in W} f_w w(g_u) \delta_{wu}.$$

Define  $y_i \in Q_{t_1, t_2}^{KK}(W)$  ( $i = 1, \dots, r$ ) by

$$y_i := A_i \delta_i + B_i \text{ where } A_i := \frac{t_1 + t_2 e^{-\alpha_i}}{1 - e^{\alpha_i}}, B_i := \frac{t_1 + t_2}{1 - e^{-\alpha_i}}.$$

**Proposition 9** *We have the following equations.*

- (1)  $(y_i - t_1)(y_i - t_2) = 0$  for  $i = 1, \dots, r$ .
- (2)  $\underbrace{y_i y_j \cdots}_{m_{i,j}} = \underbrace{y_j y_i \cdots}_{m_{i,j}}$ , where  $m_{i,j}$  is the order of  $s_i s_j$ .

**Proof:** These equations can be shown by direct calculations. □

By this proposition we can define  $y_w := y_{i_1} \cdots y_{i_\ell}$  for a reduced expression  $w = s_{i_1} \cdots s_{i_\ell}$ . These  $\{y_w\}_{w \in W}$  become a  $Q_{t_1, t_2}(\Lambda)$ -basis of  $Q_{t_1, t_2}^{KK}(W)$ .

**Remark 3** *This operator  $y_i$  can be seen as a generic Demazure-Lusztig operator. When  $t_1 = -1, t_2 = q$ , it becomes  $y_{s_i}^q$  in Kumar's book[Kum](12.2.E(9)). We can also set  $A_i$  which satisfies*

$$A_i A_{-i} = \frac{(t_1 + t_2 e^{-\alpha_i})(t_1 + t_2 e^{\alpha_i})}{(1 - e^{\alpha_i})(1 - e^{-\alpha_i})}.$$

For example, if we set  $A_i = \frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{\alpha_i}}$  and  $t_1 = q, t_2 = -1$  and replace  $\alpha_i$  by  $-\alpha_i$ , it becomes Lusztig's  $T_{s_i}$  [Lul]. If we set  $A_i = -\frac{t_1 + t_2 e^{\alpha_i}}{1 - e^{-\alpha_i}}$  and  $t_1 = -1, t_2 = v$  and replace  $\alpha_i$  by  $-\alpha_i$ , it becomes  $\mathcal{T}_i$  in [BBL].

We can define a  $Q_{t_1, t_2}(\Lambda)$ -module isomorphism  $\Phi : Q_{t_1, t_2}^{KK}(W) \rightarrow H_{t_1, t_2}^{Q(\Lambda)}(W)$  by  $\Phi(y_w) = h_w$ . Let  $\Delta_{s_i} := A_i \delta_i$ . Define  $A(w) := \prod_{\beta \in R(w)} \frac{t_1 + t_2 e^{-\beta}}{1 - e^\beta}$  and  $\Delta_w := A(w) \delta_w$ . Then it becomes that  $\Delta_{s_{i_1}} \cdots \Delta_{s_{i_\ell}} = A(w) \delta_w = \Delta_w$ . In particular,  $\Delta_{s_i}$ 's satisfy the braid relations. We can show below by induction on length  $\ell(w)$ .

**Theorem 2** *For  $w \in W$ , we have*

$$\Phi(\Delta_w) = Y_w.$$

**Proof:** If  $w = s_i$ ,  $\Delta_{s_i} = A_i \delta_i = y_i - B_i$ . Therefore  $\Phi(\Delta_{s_i}) = h_i - B_i = h_i + \frac{t_1 + t_2}{E(\alpha_i)} = Y_{s_i}$ . If  $s_i w > w$ , by induction hypothesis we can assume  $\Phi(\Delta_w) = Y_w = \sum_{u \leq w} p(u, w) h_u$ . As  $\Phi$  is a  $Q_{t_1, t_2}(\Lambda)$ -isomorphism, it follows that  $\Delta_w = \sum_{u \leq w} p(u, w) y_u$ . Then  $\Delta_{s_i w} = \Delta_{s_i} \Delta_w = A_i \delta_i \sum_{u \leq w} p(u, w) y_u =$

$\sum_{u \leq w} s_i [p(u, w)] A_i \delta_i y_u = \sum_{u \leq w} s_i [p(u, w)] (y_i - B_i) y_u = \sum_{u \leq s_i w} p(u, s_i w) y_u$ . We used the recurrence relation (Proposition 5) for the last equality. Therefore  $\Phi(\Delta_{s_i w}) = \sum_{u \leq s_i w} p(u, s_i w) h_u = Y_{s_i w}$ . The theorem is proved.  $\square$

**Corollary 2** (Explicit formula for  $\tilde{p}(w, v)$ )

Let  $v = s_{i_1} \cdots s_{i_\ell}$  be a reduced expression. Then we have

$$\tilde{p}(w, v) = \frac{1}{A(w)} \sum_{\epsilon = (\epsilon_1, \dots, \epsilon_\ell) \in \{0, 1\}^\ell, s_{i_1}^{\epsilon_1} \cdots s_{i_\ell}^{\epsilon_\ell} = w} \prod_{j=1}^{\ell} C_j(\epsilon)$$

where for  $\epsilon = (\epsilon_1, \dots, \epsilon_\ell) \in \{0, 1\}^\ell$ ,  $C_j(\epsilon) := s_{i_1}^{\epsilon_1} s_{i_2}^{\epsilon_2} \cdots s_{i_{j-1}}^{\epsilon_{j-1}} (\delta_{\epsilon_j, 1} A_{i_j} + \delta_{\epsilon_j, 0} B_{i_j})$ .

**Proof:** Taking the inverse image of the map  $\Phi$ , the equality  $h_v = \sum_{w \leq v} \tilde{p}(w, v) Y_w$  becomes

$$y_v = \sum_{w \leq v} \tilde{p}(w, v) \Delta_w = \sum_{w \leq v} \tilde{p}(w, v) A(w) \delta_w.$$

As  $v = s_{i_1} \cdots s_{i_\ell}$  is a reduced expression,  $y_v = y_{s_{i_1}} \cdots y_{s_{i_\ell}} = (A_{i_1} \delta_{i_1} + B_{i_1} \delta_e) \cdots (A_{i_\ell} \delta_{i_\ell} + B_{i_\ell} \delta_e)$ . By expanding this we get the formula.  $\square$

**Remark 4** Using Theorem 1, we also have a closed form for  $p(w, v)$ . We have another conjectural formula for  $p(w, v)$  using  $\lambda$ -chain cf. [Nar].

**Example 1** Type  $A_2$ . We use notation  $A_{-1} = \star(A_1)$ ,  $B_{-1} = \star(B_1)$ ,  $B_{12} = \frac{t_1 + t_2}{1 - e^{-(\alpha_1 + \alpha_2)}}$ .

When  $v = s_1 s_2 s_1$ ,  $w = s_1$ , then  $\epsilon = (1, 0, 0)$ ,  $(0, 0, 1)$  and

$$\tilde{p}(s_1, s_1 s_2 s_1) = (A_1 B_{12} B_{-1} + B_1 B_2 A_1) / A_1 = B_{12} B_{-1} + B_1 B_2 = B_2 B_{12}.$$

When  $v = s_1 s_2 s_1$ ,  $w = s_2$ , then  $\epsilon = (0, 1, 0)$  and

$$\tilde{p}(s_2, s_1 s_2 s_1) = (B_1 A_2 B_{12}) / A_2 = B_1 B_{12}.$$

When  $v = s_1 s_2 s_1$ ,  $w = e$ , then  $\epsilon = (0, 0, 0)$ ,  $(1, 0, 1)$  and

$$\tilde{p}(e, s_1 s_2 s_1) = B_1 B_2 B_1 + A_1 B_{12} A_{-1}.$$

## 4 Casselman's problem

In his paper [Cas] B. Casselman gave a problem concerning transition coefficients between two bases in the space of Iwahori fixed vectors of a principal series representation of a  $p$ -adic group. We relate the problem with the Yang-Baxter basis and give an answer to the problem.

### 4.1 Principal series representations of $p$ -adic group and Iwahori fixed vector

We follow the notations of M.Reeder [Re1, Re2]. Let  $G$  be a connected reductive  $p$ -adic group over a non-archimedean local field  $F$ . For simplicity we restrict to the case of split semisimple  $G$ . Associated to  $F$ , there is the ring of integer  $\mathcal{O}$ , the prime ideal  $\mathfrak{p}$  with a generator  $\varpi$ , and the residue field with  $q = |\mathcal{O}/\mathfrak{p}|$  elements. Let  $P$  be a minimal parabolic subgroup (Borel) of  $G$ , and  $A$  be the maximal split torus of  $P$  so that  $A \simeq (F^*)^r$  where  $r$  is the rank of  $G$ . For an unramified quasi-character  $\tau$  of  $A$ , i.e. a group homomorphism  $\tau : A \rightarrow \mathbb{C}^*$  which is trivial on  $A_0 = A \cap K$ , where  $K = G(\mathcal{O})$  is a maximal compact subgroup of  $G$ . Let  $T = \mathbb{C}^* \otimes X^*(A)$  be the complex torus dual to  $A$ , where  $X^*(A)$  is the group of rational characters on  $A$ , i.e.  $X^*(A) = \{\lambda : A \rightarrow F^*, \text{ algebraic group homomorphism}\}$ . We have a pairing  $\langle, \rangle : A/A_0 \times T \rightarrow \mathbb{C}^*$  given by  $\langle a, z \otimes \lambda \rangle = z^{\text{val}(\lambda(a))}$ . This gives an identification  $T \simeq X^{nr}(A)$  of  $T$  with the set of unramified quasi-characters on  $A$  (cf. [Bum] Exercise 18,19).

Let  $\Delta \subset X^*(A)$  be the set of roots of  $A$  in  $G$ ,  $\Delta^+$  be the set of positive roots corresponding to  $P$  and  $\Sigma \subset \Delta^+$  be the set of simple roots. For a root  $\alpha \in \Delta$ , we define  $e_\alpha \in X^*(T)$  by

$$e_\alpha(\tau) = \langle h_\alpha(\varpi), \tau \rangle$$

for  $\tau \in T$  where  $h_\alpha : F^* \rightarrow A$  is the one parameter subgroup (coroot) corresponding to  $\alpha$ .

**Remark 5** As the definition shows,  $e_\alpha$  is defined using the coroot  $\alpha^* = h_\alpha$ . So it should be parametrized by  $\alpha^*$ , but for convenience we follow the notation of [Re1]. Later we will identify  $e_\alpha(\alpha \in \Delta = R^*)$  with  $e^\alpha(\alpha \in R = \Delta^*)$  by the map  $*$  :  $\Delta \rightarrow R$  of root data.

$W$  acts on right of  $X^{nr}(A)$  so that  $\tau^w(a) = \tau(waw^{-1})$  for  $a \in A$ ,  $\tau \in T$  and  $w \in W$ . The action of  $W$  on  $X^*(T)$  is given by  $(we_\alpha)(\tau) = e_{w\alpha}(\tau) = e_\alpha(\tau^w)$  for  $\alpha \in \Delta$ ,  $\tau \in T$  and  $w \in W$ .

The principal series representation  $I(\tau)$  of  $G$  associated to an unramified quasicharacter  $\tau$  of  $A$  is defined as follows. As a vector space over  $\mathbb{C}$  it consists of locally constant functions on  $G$  with values in  $\mathbb{C}$  which satisfy the left relative invariance properties with respect to  $P$  where  $\tau$  is extended to  $P$  with trivial value on the unipotent radical  $N$  of  $P = AN$ .

$$I(\tau) := \text{Ind}_P^G(\tau) = \{f : G \rightarrow \mathbb{C} \text{ loc. const. function} \mid f(pg) = \tau\delta^{1/2}(p)f(g) \text{ for } \forall p \in P, \forall g \in G\}.$$

Here  $\delta$  is the modulus of  $P$ . The action of  $G$  on  $I(\tau)$  is defined by right translation, i.e. for  $g \in G$  and  $f \in I(\tau)$ ,  $(\pi(g)f)(x) = f(xg)$ .

Let  $B$  be the Iwahori subgroup which is the inverse image  $\pi^{-1}(P(\mathbb{F}_q))$  of the Borel subgroup  $P(\mathbb{F}_q)$  of  $G(\mathbb{F}_q)$  by the projection  $\pi : G(\mathcal{O}) \rightarrow G(\mathbb{F}_q)$ . Then we define  $I(\tau)^B$  to be the space of Iwahori fixed vectors in  $I(\tau)$ , i.e.

$$I(\tau)^B := \{f \in I(\tau) \mid f(gb) = f(g) \text{ for } \forall b \in B, \forall g \in G\}.$$

This space has a natural basis  $\{\varphi_w^\tau\}_{w \in W}$ .  $\varphi_w^\tau \in I(\tau)^B$  is supported on  $PwB$  and satisfies

$$\varphi_w^\tau(pwb) = \tau\delta^{1/2}(p) \text{ for } \forall p \in P, \forall b \in B.$$

### 4.2 Intertwiner and Casselman's basis

From now on we always assume that  $\tau$  is regular i.e. the stabilizer  $W_\tau = \{w \in W \mid \tau^w = \tau\}$  is trivial. The intertwining operator  $\mathcal{A}_w^\tau : I(\tau) \rightarrow I(\tau^w)$  is defined by

$$\mathcal{A}_w^\tau(f)(g) := \int_{N_w} f(wng)dn$$



where  $N_w := N \cap w^{-1}N_-w$ , with  $N_-$  being the unipotent radical of opposite parabolic  $P_-$  which corresponds to the negative roots  $\Delta^-$ . The integral is convergent when  $|e_\alpha(\tau)| < 1$  for all  $\alpha \in \Delta^+$  such that  $w\alpha \in \Delta^-$  (cf. [Bum] Proposition 63), and may be meromorphically continued to all  $\tau$ . It has the property that for  $x, y \in W$  with  $\ell(xy) = \ell(x) + \ell(y)$ , then

$$\mathcal{A}_y^\tau \mathcal{A}_x^\tau = \mathcal{A}_{xy}^\tau.$$

The Casselman's basis  $\{f_w^\tau\}_{w \in W}$  of  $I(\tau)^B$  is defined as follows.  $f_w^\tau \in I(\tau)^B$  and

$$\mathcal{A}_y^\tau f_w^\tau(1) = \begin{cases} 1 & \text{if } y = w \\ 0 & \text{if } y \neq w. \end{cases}$$

M.Reeder characterizes this using the action of affine Hecke algebra (cf. [Re2] Section 2). The affine Hecke algebra  $\mathcal{H} = \mathcal{H}(G, B)$  is the convolution algebra of  $B$  bi-invariant locally constant functions on  $G$  with values in  $\mathbb{C}$ . By the theorem of Iwahori-Matsumoto it can be described by generators and relations. The basis  $\{T_w\}_{w \in \tilde{W}_{aff}}$  consists of characteristic functions  $T_w := ch_{BwB}$  of double coset  $BwB$ . Let  $\mathcal{H}_W$  be the Hecke algebra of the finite Weyl group  $W$  generated by the simple reflections  $s_\alpha$  for simple roots  $\alpha \in \Sigma$ . As a vector space  $\mathcal{H}$  is the tensor product of two subalgebras  $\mathcal{H} = \Theta \otimes \mathcal{H}_W$ . The subalgebra  $\Theta$  is commutative and isomorphic to the coordinate ring of the complex torus  $T$  with a basis  $\{\theta_a \mid a \in A/A_0\}$ , where  $\theta_a$  is defined as follows (cf. [Lu2]). Define  $A^- := \{a \in A \mid |\alpha(a)|_F \leq 1 \forall \alpha \in \Sigma\}$ . For  $a \in A$ , choose  $a_1, a_2 \in A^-$  such that  $a = a_1 a_2^{-1}$ . Then  $\theta_a = q^{(\ell(a_1) - \ell(a_2))/2} T_{a_1} T_{a_2}^{-1}$  where for  $x \in G$ ,  $\ell(x)$  is the length function defined by  $q^{\ell(x)} = [BxB : B]$  and  $T_x \in \mathcal{H}$  is the characteristic function of  $BxB$ .

By Lemma (4.1) of [Re1], there exists a unique  $f_w^\tau \in I(\tau)_w \cap I(\tau)^B$  for each  $w \in W$  such that

- (1)  $f_w^\tau(w) = 1$  and
- (2)  $\pi(\theta_a) f_w^\tau = \tau^w(a) f_w^\tau$  for all  $a \in A$ .

Here  $I(\tau)_w := \{f \in I(\tau) \mid \text{support of } f \text{ is contained in } \bigcup_{x \geq w} P_x P\}$ .

### 4.3 Transition coefficients

Let

$$f_w^\tau = \sum_{w \leq v} a_{w,v}(\tau) \varphi_v^\tau$$

and

$$\varphi_w^\tau = \sum_{w \leq v} b_{w,v}(\tau) f_v^\tau.$$

The Casselman's problem is to find an explicit formula for  $a_{w,v}(\tau)$  and  $b_{w,v}(\tau)$ .

To relate the results in Sections 2 and 3 with the Casselman's problem, in this subsection we specialize the parameters  $t_1 = -q^{-1}$ ,  $t_2 = 1$  and take tensor product with the complex field  $\mathbb{C}$ . For example, the Yang-Baxter basis  $Y_w$  will become a  $Q_{t_1, t_2}(\Lambda) \otimes \mathbb{C}$  basis in  $H_{t_1, t_2}^{Q(\Lambda)}(W)_{\mathbb{C}} = H_{t_1, t_2}^{Q(\Lambda)}(W) \otimes \mathbb{C}$ . The generic Demazure-Lusztig operator defined in Section 3 will become

$$y_i := A_i \delta_i + B_i \text{ where } A_i := \frac{-q^{-1} + e^{-\alpha_i}}{1 - e^{\alpha_i}}, B_i := \frac{-q^{-1} + 1}{1 - e^{-\alpha_i}}.$$

Then  $(y_i + q^{-1})(y_i - 1) = 0$ .

**Theorem 3** We identify  $e^\alpha$  with  $e_\alpha$  (cf. Remark 5). Then,

$$a_{w,v}(\tau) = \tilde{p}(w, v)(\tau)|_{t_1=-q^{-1}, t_2=1}$$

$$b_{w,v}(\tau) = p(w, v)(\tau)|_{t_1=-q^{-1}, t_2=1}.$$

**Proof:**  $b_{w,v}$ 's satisfy the same recurrence relation (Proposition 5 with  $t_1 = -q^{-1}, t_2 = 1$ ) as  $p(w, v)$ 's (cf. [Re2] Proposition (2.2)). The initial condition  $b_{w,w} = p(w, w) = 1$  leads to the second equation. The first equation then also holds. Note that the  $b_{y,w}$  in [Re2] is our  $b_{w,y}$ . □

**Remark 6** There is also a direct proof that does not use recurrence relation cf. [NN].

**Corollary 3** We have a closed formula for  $a_{w,v}(\tau)$  and  $b_{w,v}(\tau)$  by Corollary 2 and Theorem 1.

**Corollary 4** For  $v \in W$ , we have

$$\sum_{w \leq v} b_{w,v} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}e^\beta}{1 - e^\beta},$$

and

$$\sum_{w \leq v} b_{w,v}(-q^{-1})^{\ell(w)} = \prod_{\beta \in R(v)} \frac{1 - q^{-1}}{1 - e^\beta}.$$

**Proof:** When  $t_1 = -q^{-1}, t_2 = 1$ , we can specialize  $h_i$  to 1 and we get the first equation from the definition of  $Y_v$ , since  $1 + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}e^\beta}{1-e^\beta}$ . We can also specialize  $h_i$  to  $-q^{-1}$  and  $-q^{-1} + \frac{(1-q^{-1})e^\beta}{1-e^\beta} = \frac{1-q^{-1}}{1-e^\beta}$  gives the second equation. □

**Remark 7** The left hand side of the first equation in Corollary 4 is  $m(e, v^{-1})$  in [BN]. So this gives another proof of Theorem 1.4 in [BN].

### 4.4 Whittaker function

M.Reeder [Re2] specified a formula for the Whittaker function  $\mathcal{W}_\tau(f_w^\tau)$  and using  $b_{w,v}$ , he got a formula for  $\mathcal{W}_\tau(\varphi_w^\tau)$ . For  $a \in A$ , let  $\lambda_a \in X^*(T)$  be

$$\lambda_a(z \otimes \mu) = z^{val(\mu(a))} \text{ for } z \in \mathbb{C}^*, \mu \in X^*(A).$$

Formally the result of M.Reeder [Re2] Corollary (3.2) is written as follows. For  $w \in W$  and  $a \in A^-$ ,

$$\mathcal{W}(\varphi_w)(a) = \delta^{1/2}(a) \sum_{w \leq y} b_{w,y} y \left[ \lambda_a \prod_{\beta \in R^+ - R(y)} \frac{1 - q^{-1}e^\beta}{1 - e^{-\beta}} \right] \in \mathbb{C}[T].$$

Then using Corollary 3, we have an explicit formula of  $\mathcal{W}(\varphi_w)(a)$ .

### 4.5 Relation with Bump-Nakasuji's work

Now we explain the relation between this paper and Bump-Nakasuji [BN]. First of all, the notational conventions are slightly different. Especially in the published [BN] the natural base and intertwiner are differently parametrized. The natural basis  $\phi_w$  in [BN] is our  $\varphi_{w^{-1}}$ . The intertwiner  $M_w$  in [BN] is our  $\mathcal{A}_{w^{-1}}$  so that if  $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$ ,  $M_{w_1w_2} = M_{w_1} \circ M_{w_2}$  while  $\mathcal{A}_{w_1w_2} = \mathcal{A}_{w_2}\mathcal{A}_{w_1}$ .

In the paper [BN], another basis  $\{\psi_w\}_{w \in W}$  for the space  $I(\tau)^B$  was defined and compared with the Casselman's basis. They defined  $\psi_w := \sum_{v \geq w} \varphi_v$  and expand this as  $\psi_w = \sum_{v \geq w} m(w, v)f_v$  and conversely  $f_w = \sum_{v \geq w} \tilde{m}(w, v)\psi_v$ . They observed that the transition coefficients  $m(w, v)$  and  $\tilde{m}(w, v)$  factor under certain condition. Let  $S(w, v) := \{\alpha \in R^+ | w \leq s_\alpha v < v\}$  and  $S'(w, v) := \{\alpha \in R^+ | w < s_\alpha w \leq v\}$ . Then the statements of the conjectures are as follows.

**Conjecture 1** ([BN] Conjecture 1.2) Assume that the root system  $R$  is simply-laced. Suppose  $w \leq v$  and  $|S(w, v)| = \ell(v) - \ell(w)$ , then

$$m(w, v) = \prod_{\alpha \in S(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$

**Conjecture 2** ([BN] Conjecture 1.3) Assume that the root system  $R$  is simply-laced. Suppose  $w \leq v$  and  $|S'(w, v)| = \ell(v) - \ell(w)$ , then

$$\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} \prod_{\alpha \in S'(w, v)} \frac{1 - q^{-1}z^\alpha}{1 - z^\alpha}.$$

**Proposition 10** Conjecture 1.2 and Conjecture 1.3 in [BN] are equivalent.

**Proof:** We can show  $m(w, v) = \sum_{w \leq z \leq v} p(z, v)$  and  $\tilde{m}(w, v) = \sum_{w \leq z \leq v} (-1)^{\ell(v) - \ell(z)} \tilde{p}(w, z)$ . Then it follows by the Theorem 1 that  $\tilde{m}(w, v) = (-1)^{\ell(v) - \ell(w)} m(vw_0, ww_0)$ . As  $S'(w, v) = S(vw_0, ww_0)$  we get the desired conclusion.  $\square$

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