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$\mathrm{GL}_n(\mathbf{F}_q)$-*analogues of factorization problems in* $\mathfrak{S}_n$

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Abstract. We consider $\mathrm{GL}_n(\mathbf{F}_q)$-analogues of certain factorization problems in the symmetric group $\mathfrak{S}_n$: rather than counting factorizations of the long cycle $(1, 2, \ldots, n)$ given the number of cycles of each factor, we count factorizations of a regular elliptic element given the fixed space dimension of each factor. We show that, as in $\mathfrak{S}_n$, the generating function counting these factorizations has attractive coefficients after an appropriate change of basis. Our work generalizes several recent results on factorizations in $\mathrm{GL}_n(\mathbf{F}_q)$ and also uses a character-based approach. We end with an asymptotic application and some questions.

Résumé. Nous considérons $\mathrm{GL}_n(\mathbf{F}_q)$-analogues de certains problèmes de factorisation dans le groupe symétrique $\mathfrak{S}_n$: plutôt que de compter factorisations d'un grand cycle $(1, 2, \ldots, n)$ étant donné le nombre de cycles de chaque facteur, nous comptons factorisations de un élément elliptique régulière donné la dimension de l'espace fixe de chaque facteur. Nous montrons que, comme dans $\mathfrak{S}_n$, la fonction génératrice de ces factorisations a des coefficients attirants après un changement de base. Notre travail généralise plusieurs résultats récents sur factorisations dans $\mathrm{GL}_n(\mathbf{F}_q)$ et utilise une approche basée sur les caractères. Nous terminons avec une application asymptotique et des questions.

Keywords. factorization, finite general linear group, Singer cycle, regular elliptic, fixed space dimension, q-analogue

1 Introduction

There is a rich vein in combinatorics of problems related to factorizations in the symmetric group $\mathfrak{S}_n$. Frequently, the size of a certain family of factorizations is unwieldy but has an attractive generating function, possibly after an appropriate change of basis. As a prototypical example, one might seek to count factorizations $c = u \cdot v$ of the **long cycle** $c = (1, 2, \ldots, n)$ in $\mathfrak{S}_n$ as a product of two permutations, keeping track of the *number of cycles* or even the *cycle types* of the two factors. Notably, results of this form have been given by Jackson [11, §4], [12], including the following result.

Theorem 1.1 (Jackson [12]; Morales–Vassilieva [15]). *Let $a_{r,s}$ be the number of pairs (u, v) of elements of $\mathfrak{S}_n$ such that u has r cycles, v has s cycles, and $c = u \cdot v$. Then*

$$\frac{1}{n!} \sum_{r,s \geq 0} a_{r,s} \cdot x^r y^s = \sum_{t,u \geq 1} \binom{n-1}{t-1; u-1; n-t-u+1} \binom{x}{t} \binom{y}{u}. \tag{1.1}$$

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Moreover, for λ, μ partitions of n, let $a_{\lambda,\mu}$ be the number of pairs (u, v) of elements of $\mathfrak{S}_n$ such that u has cycle type λ, v has cycle type μ, and $c = u \cdot v$. Then

$$\frac{1}{n!} \sum_{\lambda,\mu\vdash n} a_{\lambda,\mu} \cdot p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) = \sum_{\alpha,\beta} \frac{(n-\ell(\alpha))!(n-\ell(\beta))!}{(n-1)!(n+1-\ell(\alpha)-\ell(\beta))!} \mathbf{x}^\alpha \mathbf{y}^\beta, \tag{1.2}$$

where p_λ denotes the usual power-sum symmetric function and the sum on the right is over all weak compositions α, β of n.

Recently, there has been interest in q-analogues of such problems, replacing $\mathfrak{S}_n$ with the general linear group $\mathrm{GL}_n(\mathbf{F}_q)$ over an arbitrary finite field $\mathbf{F}_q$, the long cycle with a *Singer cycle* (or, more generally, *regular elliptic element*) c, and the number of cycles with the *fixed space dimension* [14, 10]. In the present paper, we extend this approach to give the following q-analogue of Theorem 1.1. Our theorem statement uses the standard notations

$$(a;q)_m = (1-a)(1-aq)\cdots(1-aq^{m-1}) \quad \text{and} \quad [m]!_q = \frac{(q;q)_m}{(1-q)^m} = 1{\cdot}(1+q)\cdots(1+q+\ldots+q^{m-1}).$$

Theorem 1.2. *Fix a regular elliptic element c in $G = \mathrm{GL}_n(\mathbf{F}_q)$. Let $a_{r,s}(q)$ be the number of pairs (u, v) of elements of G such that u has fixed space dimension r, v has fixed space dimension s, and $c = u \cdot v$. Then*

$$\frac{1}{|G|} \sum_{r,s\geq 0} a_{r,s}(q) \cdot x^r y^s = \frac{(x;q^{-1})_n}{(q;q)_n} + \frac{(y;q^{-1})_n}{(q;q)_n} + \\ \sum_{\substack{0\leq t,u\leq n-1 \\ t+u\leq n}} q^{tu-t-u} \frac{[n-t-1]!_q \cdot [n-u-1]!_q}{[n-1]!_q \cdot [n-t-u]!_q} \frac{(q^n-q^t-q^u+1)}{(q-1)} \cdot \frac{(x;q^{-1})_t}{(q;q)_t} \frac{(y;q^{-1})_u}{(q;q)_u}. \tag{1.3}$$

More generally, in either $\mathfrak{S}_n$ or $\mathrm{GL}_n(\mathbf{F}_q)$ one may consider factorizations into more than two factors. In $\mathfrak{S}_n$, this gives the following result.

Theorem 1.3 (Jackson [12]; Bernardi–Morales [2])**.** *Let $a_{r_1,r_2,\ldots,r_k}$ be the number of tuples $(u_1, u_2, \ldots, u_k)$ of permutations in $\mathfrak{S}_n$ such that u_i has r_i cycles and $u_1u_2\cdots u_k = c$. Then*

$$\frac{1}{(n!)^{k-1}} \sum_{1\leq r_1,r_2,\ldots,r_k\leq n} a_{r_1,\ldots,r_k} \cdot x_1^{r_1}\cdots x_k^{r_k} = \sum_{1\leq p_1,\ldots,p_k\leq n} M^{n-1}_{p_1-1,\ldots,p_k-1} \binom{x_1}{p_1}\cdots\binom{x_k}{p_k}, \tag{1.4}$$

where

$$M^m_{r_1,\ldots,r_k} := \sum_{d=0}^{\min(r_i)} (-1)^d \binom{m}{d} \prod_{i=1}^k \binom{m-d}{r_i-d}. \tag{1.5}$$

Moreover, let $a_{\lambda^{(1)},\ldots,\lambda^{(k)}}$ be the number of tuples $(u_1, u_2, \ldots, u_k)$ of permutations in $\mathfrak{S}_n$ such that u_i has cycle type $\lambda^{(i)}$ and $u_1u_2\cdots u_k = c$. Then

$$(n!)^{1-k} \sum_{\lambda^{(1)},\cdots,\lambda^{(k)}\vdash n} a_{\lambda^{(1)},\ldots,\lambda^{(k)}} \cdot p_{\lambda^{(1)}}(\mathbf{x}_1)\cdots p_{\lambda^{(k)}}(\mathbf{x}_k) = \sum_{\alpha^{(1)},\ldots,\alpha^{(k)}} \frac{M^{n-1}_{\ell(\alpha^{(1)})-1,\ldots,\ell(\alpha^{(k)})-1}}{\prod_{i=1}^k \binom{n-1}{\ell(\alpha^{(i)})-1}} (\mathbf{x}_1)^{\alpha^{(1)}}\cdots(\mathbf{x}_k)^{\alpha^{(k)}},$$

where the sum on the right is over all weak compositions $\alpha^{(1)}, \ldots, \alpha^{(k)}$ of n.

In the present paper, we prove the following q-analogue of this result. The statement uses the standard q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]!_q \big/ ([k]!_q \cdot [n-k]!_q)$.

Theorem 1.4. *Fix a regular elliptic element c in $G = \mathrm{GL}_n(\mathbf{F}_q)$. Let $a_{r_1,\ldots,r_k}(q)$ be the number of tuples $(u_1, \ldots, u_k)$ of elements of G such that u_i has fixed space dimension r_i and $u_1 \cdots u_k = c$. Then*

$$\frac{1}{|G|^{k-1}} \sum_{r_1,\ldots,r_k} a_{r_1,\ldots,r_k}(q) \cdot x_1^{r_1} \cdots x_k^{r_k} = \sum_{\substack{\mathbf{p}=(p_1,\ldots,p_k):\\ 0 \le p_i \le n}} \frac{M_{\widetilde{\mathbf{p}}}^{n-1}(q)}{\prod_{p \in \widetilde{\mathbf{p}}} \begin{bmatrix} n-1 \\ p \end{bmatrix}_q} \cdot \frac{(x_1; q^{-1})_{p_1}}{(q;q)_{p_1}} \cdots \frac{(x_k; q^{-1})_{p_k}}{(q;q)_{p_k}}, \tag{1.6}$$

where $\widetilde{\mathbf{p}}$ is the result of deleting all copies of n from $\mathbf{p}$,

$$M_{r_1,\ldots,r_k}^{m}(q) := \sum_{d=0}^{\min(r_i)} (-1)^d q^{\binom{d+1}{2} - kd} \begin{bmatrix} m \\ d \end{bmatrix}_q \prod_{i=1}^{k} \begin{bmatrix} m-d \\ r_i - d \end{bmatrix}_q \tag{1.7}$$

for $k > 0$, and $M_{\varnothing}^m(q) := 0$.

Remark 1.5. In viewing Theorems 1.2 and 1.4 as q-analogues of Theorems 1.1 and 1.3, it is helpful to first observe that if $x = q^N$ is a positive integer power of q then $\dfrac{(x; q^{-1})_m}{(q;q)_m} = \begin{bmatrix} N \\ m \end{bmatrix}_q$. Further, we have the equality

$$\lim_{q \to 1} \frac{[n-t-1]!_q \cdot [n-u-1]!_q}{[n-1]!_q \cdot [n-t-u]!_q} \frac{(q^n - q^t - q^u + 1)}{(q-1)} = \frac{(n-(t+1))!(n-(u+1))!}{(n-1)!(n+1-(t+1)-(u+1))!}$$

between the limit of a coefficient in (1.3) and a coefficient on the right side of (1.2), and more generally the equality $\lim_{q\to 1} M_{r_1,\ldots,r_k}^m(q) = M_{r_1,\ldots,r_k}^m$.

Note that the generating function (1.6) is (in its definition) analogous to the less-refined generating function (1.4), while the coefficient

$$M_{\widetilde{\mathbf{p}}}^{n-1}(q) \Big/ \prod_{p \in \widetilde{\mathbf{p}}} \begin{bmatrix} n-1 \\ p \end{bmatrix}_q$$

is analogous (in the $q \to 1$ sense) to a coefficient in the more refined half of Theorem 1.3. This phenomenon is mysterious. A similar phenomenon was observed in the discussion following Theorem 4.2 in [10], namely, that the counting formula $q^{e(\alpha)}(q^n - 1)^{k-1}$ for factorizations of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ into k factors with fixed space codimensions given by the composition α of n is a q-analogue of the counting formula n^{k-1} for factorizations of an n-cycle as a genus-0 product of k *cycles* of specified lengths.

On the other hand, we *can* give a heuristic explanation for the fact that the lower indices in the M-coefficients in Theorem 1.4 are shifted by 1 compared with those in Theorem 1.3: the matrix group $\mathfrak{S}_n$ does not act irreducibly in its standard representation, as every permutation fixes the all-ones vector. Thus, morally, the subtraction of 1 should correct for the irrelevant dimension of fixed space.

Our approach is to follow a well-worn path, based on character-theoretic techniques that go back to Frobenius. In the case of the symmetric group, this approach has been extensively developed in the '80s and '90s, notably in work of Stanley [19], Jackson [11, 12], and Goupil–Schaeffer [7]. In $\mathrm{GL}_n(\mathbf{F}_q)$, the necessary character theory was worked out by Green [8]. This approach has been used recently by the first-named author and coauthors to count factorizations of Singer cycles into reflections [14] and to count *genus*-0 factorizations (that is, those in which the codimensions of the fixed spaces of the factors sum to the codimension of the fixed space of the product) of regular elliptic elements [10]. The current work subsumes these previous results while requiring no new character values. (See Remarks 3.2 and 3.3 for derivations of these earlier results from Theorem 1.4.)

The plan of the paper is as follows. In Section 2, we provide background, including an overview of the character-theoretic approach to problems of this sort and a quick introduction to the character theory of $\mathrm{GL}_n(\mathbf{F}_q)$ necessary for our purposes. Proofs of Theorems 1.2 and 1.4 are sketched in Section 3. The **genus** of a factorization counted in $a_{r,s}(q)$ is $n - r - s$. In Section 4, we give an application of Theorem 1.2 to asymptotic enumeration, giving the precise growth rate $\Theta(q^{(n+g)^2/2}/|\mathrm{GL}_g(\mathbf{F}_q)|)$ of the number of factorizations of fixed genus g of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ as a product of two factors, as $n \to \infty$. Finally, in Section 5 we give a few additional remarks and open questions.

The full-length version of this extended abstract is available as [13].

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2 Regular elliptics, character theory, and the symmetric group approach

2.1 Singer cycles and regular elliptic elements

The field $\mathbf{F}_{q^n}$ is an n-dimensional vector space over $\mathbf{F}_q$, and multiplication by a fixed element in the larger field is a linear transformation. Thus, any choice of basis for $\mathbf{F}_{q^n}$ over $\mathbf{F}_q$ gives a natural inclusion $\mathbf{F}_{q^n}^\times \hookrightarrow G_n := \mathrm{GL}_n(\mathbf{F}_q)$. The image of any cyclic generator c for $\mathbf{F}_{q^n}^\times$ under this inclusion is called a **Singer cycle**. A strong analogy between Singer cycles in G_n and n-cycles in $\mathfrak{S}_n$ has been established over the past decade or so, notably in work of Reiner, Stanton, and collaborators [17, 16, 14, 10]. As one elementary example of this analogy, the Singer cycles act transitively on the lines of $\mathbf{F}_q^n$, just as the n-cycles act transitively on the points $\{1, \ldots, n\}$.

A more general class of elements of G_n, containing the Singer cycles, is the set of images (under the same inclusion) of field generators σ for $\mathbf{F}_{q^n}$ over $\mathbf{F}_q$. (That is, one should have $\mathbf{F}_q[\sigma] = \mathbf{F}_{q^n}$ but not necessarily $\{\sigma^m \mid m \in \mathbb{Z}\} = \mathbf{F}_{q^n}^\times$.) Such elements are called **regular elliptic elements**. They may be characterized in several other ways; see [14, Prop. 4.4].

2.2 The character-theoretic approach to factorization problems

Given a finite group G, let $\mathrm{Irr}(G)$ be the collection of its irreducible (finite-dimensional, complex) representations V. For each V in $\mathrm{Irr}(G)$, denote by $\deg(V) := \dim_{\mathbb{C}} V$ its **degree**, by $\chi^V(g) := \mathrm{Tr}(g : V \to V)$ its **character value** at g, and by $\widetilde{\chi}^V(g) := \chi^V(g)/\deg(V)$ its **normalized character value**. The functions $\chi^V(-)$ and $\widetilde{\chi}^V(-)$ on G extend by $\mathbb{C}$-linearity to functionals on the group algebra $\mathbb{C}[G]$.

The following result allows one to express every factorization problem of the form we consider as a computation in terms of group characters.

Proposition 2.1 (Frobenius [4]). *Let G be a finite group, and $A_1, \ldots, A_\ell \subseteq G$ unions of conjugacy classes in G. Then for g in G, the number of ordered factorizations $(t_1, \ldots, t_\ell)$ with $g = t_1 \cdots t_\ell$ and t_i in A_i for $i = 1, 2, \ldots, \ell$ is*

$$\frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(g^{-1}) \cdot \widetilde{\chi}^V(z_1) \cdots \widetilde{\chi}^V(z_\ell), \tag{2.1}$$

where $z_i := \sum_{t \in A_i} t$ in $\mathbb{C}[G]$.

In practice, it is often the case that one does not need the full set of character values that appear in (2.1) in order to evaluate the sum. As an example of this phenomenon, we show how to derive Theorem 1.1 without needing access to the full character table for the symmetric group $\mathfrak{S}_n$. This argument also provides a template for our work in $\mathrm{GL}_n(\mathbf{F}_q)$.

Let c be the long cycle $c = (1, 2, \ldots, n)$ in $\mathfrak{S}_n$. Consider the generating function

$$F(x, y) = \sum_{1 \le r, s \le n} a_{r,s} \cdot x^r y^s \tag{2.2}$$

for the number $a_{r,s}$ of factorizations $c = u \cdot v$ in which u, v have r, s cycles, respectively. By Proposition 2.1, we have that

$$a_{r,s} = \frac{1}{n!} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot \widetilde{\chi}^V(z_r) \widetilde{\chi}^V(z_s),$$

where z_i is the formal sum in $\mathbb{C}[\mathfrak{S}_n]$ of all elements with i cycles. Substituting this in (2.2) gives

$$\begin{aligned} F(x, y) &= \frac{1}{n!} \sum_{1 \le r, s \le n} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot \widetilde{\chi}^V(z_r) \widetilde{\chi}^V(z_s) \cdot x^r y^s \\ &= \frac{1}{n!} \sum_{V \in \mathrm{Irr}(\mathfrak{S}_n)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y), \end{aligned}$$

where $f_V(x) := \sum_{r=1}^n \widetilde{\chi}^V(z_r) x^r$. The irreducible representations of $\mathfrak{S}_n$ are indexed by partitions λ of n, and we write $\chi^V = \chi^\lambda$ if V is indexed by λ. The degree of a character is given by the *hook-length formula*. It follows from the *Murnaghan–Nakayama rule* that the character value $\chi^\lambda(c^{-1})$ on the n-cycle c^{-1} is equal to 0 unless $\lambda = \langle n-d, 1^d \rangle$ is a **hook shape**, in which case $\chi^{\langle n-d, 1^d \rangle}(c^{-1}) = (-1)^d$. Thus it suffices to understand $f_\lambda(x)$ for hooks λ. One can show that

$$f_{\langle n-d, 1^d \rangle}(x) = (x-d)_n := (x-d) \cdot (x-d+1) \cdots (x-d+n-1) = n! \cdot \sum_{k=d+1}^{n} \binom{n-1-d}{k-1-d} \cdot \binom{x}{k},$$

and the result follows by identities for binomial coefficients after extracting the coefficient of $\binom{x}{t}\binom{y}{u}$.

2.3 Character theory of the finite general linear group

In this section, we give a (very) brief overview of the character theory of $G_n = \mathrm{GL}_n(\mathbf{F}_q)$, including the specific character values necessary to prove the main results in this paper. For a proper treatment, see [21, Ch. 3] or [9, §4]. Throughout this section, we freely conflate the (complex, finite-dimensional) representation V for $\mathrm{GL}_n(\mathbf{F}_q)$ with its character χ^V.

The basic building-block of the character theory of G_n is **parabolic** (or **Harish-Chandra**) **induction**, defined as follows. For nonnegative integers a, b, let $P_{a,b}$ be the parabolic subgroup

$$P_{a,b} = \left\{ \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : A \in G_a, B \in G_b, \text{ and } C \in \mathbf{F}_q^{a\times b} \right\}$$

of G_{a+b}. Given characters χ_1 and χ_2 for G_a and G_b, respectively, one obtains a character $\chi_1 * \chi_2$ for G_{a+b} by the formula

$$(\chi_1 * \chi_2)(g) = \frac{1}{|P_{a,b}|} \sum_{\substack{h\in G_{a+b}: \\ hgh^{-1}\in P_{a,b}}} \chi_1(A)\chi_2(B),$$

where A and B are the diagonal blocks of hgh^{-1} as above.

Many irreducible characters for G_n may be obtained as irreducible components of induction products of characters on smaller general linear groups. A character $\mathcal{C}$ for G_n that cannot be so-obtained is called **cuspidal**, of **weight** $\mathrm{wt}(\mathcal{C}) = n$. The set of cuspidals for G_n is denoted Cusp_n, and the set of all cuspidals for all general linear groups is denoted $\mathrm{Cusp} = \sqcup_{n\geq 1}\mathrm{Cusp}_n$. (Though we will not need this, we note that cuspidals may be indexed by irreducible polynomials over $\mathbf{F}_q$, or equivalently by primitive q-colored necklaces.)

Let Par denote the set of all integer partitions. The set of *all* irreducible characters for G_n is indexed by functions $\underline{\lambda}$: Cusp $\to$ Par such that

$$\sum_{\mathcal{C}\in\mathrm{Cusp}} \mathrm{wt}(\mathcal{C}) \cdot |\underline{\lambda}(\mathcal{C})| = n.$$

(In particular, $\underline{\lambda}(\mathcal{C})$ must be equal to the empty partition for all but finitely many choices of $\mathcal{C}$.) A particular representation of interest is the trivial representation $\mathbf{1}$ for $\mathrm{GL}_1(\mathbf{F}_q)$. (The trivial representation for $\mathrm{GL}_n(\mathbf{F}_q)$ is indexed by the function associating to $\mathbf{1}$ the partition $\langle n\rangle$.) If V is indexed by $\underline{\lambda}$ having support on a single cuspidal representation $\mathcal{C}$, we call V **primary** and denote it by the pair $(\mathcal{C}, \lambda)$ where $\lambda = \underline{\lambda}(\mathcal{C})$.

A priori, in order to use Proposition 2.1 in our setting, we require the degrees and certain other values of all irreducible characters for G_n. In fact, however, we will only need a very small selection of them. The character degrees were worked out by Steinberg [20] and Green [8], and the special case relevant to our work is

$$\deg\left(\chi^{\mathbf{1},\langle n-d,1^d\rangle}\right) = q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q. \tag{2.3}$$

The relevant character values on regular elliptic elements were computed by Lewis–Reiner–Stanton.

Proposition 2.2 ([14, Prop. 4.7])**.** *Suppose c is a regular elliptic element and $\chi^{\underline{\lambda}}$ an irreducible character of G_n.*

(i) One has $\chi^{\underline{\lambda}}(c) = 0$ *unless* $\chi^{\underline{\lambda}}$ *is a primary irreducible character* $\chi^{U,\lambda}$ *for some* s *dividing* n *and some cuspidal character* U *in* Cusp_s*, and* $\lambda = \left\langle \frac{n}{s} - d, 1^d \right\rangle$ *is a hook-shaped partition of* n/s.

(ii) If $U = \mathbf{1}$ *is the trivial character then* $\chi^{\mathbf{1},\left\langle n-d,1^d \right\rangle}(c) = (-1)^d$.

Finally, denote by z_k the formal sum (in $\mathbb{C}[G_n]$) of all elements of G_n having fixed space dimension equal to k. Huang–Lewis–Reiner computed the relevant character values on the z_k.

Proposition 2.3 ([10, Prop. 4.10])**.** *(i) For any* s *dividing* n*, any cuspidal representation* U *in* Cusp_s *other than* $\mathbf{1}$*, and any partition* λ *of* $\frac{n}{s}$*, we have* $\widetilde{\chi}^{U,\lambda}(z_r) = (-1)^{n-r} q^{\binom{n-r}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_q$.

(ii) For $U = \mathbf{1}$ *and* $\lambda = \left\langle n-d, 1^d \right\rangle$ *a hook, we have*

$$\widetilde{\chi}^{\mathbf{1},\left\langle n-d,1^d \right\rangle}(z_r) =$$
$$(-1)^{n-r} q^{\binom{n-r}{2}} \Big(\begin{bmatrix} n \\ r \end{bmatrix}_q + \frac{(1-q)[n]_q}{[r]!_q} \cdot \sum_{j=1}^{n-\max(r,d)} q^{jr-d} \cdot \frac{[n-j]!_q}{[n-r-j]!_q} \cdot (q^{n-d-j+1}; q)_{j-1} \Big).$$

3 Proofs of main theorems

3.1 Proof sketch of Theorem 1.2

In this section, we prove Theorem 1.2 by following the approach for $\mathfrak{S}_n$ sketched in Section 2.2. Let c be a regular elliptic element in $G = \mathrm{GL}_n(\mathbf{F}_q)$, and let $a_{r,s}(q)$ be the number of pairs (u, v) of elements of G such that $u \cdot v = c$ and u, v have fixed space dimensions r, s, resprectively. Define the generating function $F(x, y) := \sum_{r,s \geq 0} a_{r,s}(q) x^r y^s$. Our goal is to rewrite this generating function in the basis $\frac{(x;q^{-1})_t}{(q;q)_t} \frac{(y;q^{-1})_u}{(q;q)_u}$ of polynomials q-analogous to the binomial coefficients.

By Proposition 2.1, we may write

$$a_{r,s}(q) = \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot \widetilde{\chi}^V(z_r) \cdot \widetilde{\chi}^V(z_s) \tag{3.1}$$

where z_k is defined (as above) to be the element of the group algebra $\mathbb{C}[G]$ equal to the sum of all elements of fixed space dimension k. Thus, our generating function is given by

$$F(x, y) = \frac{1}{|G|} \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) \cdot f_V(y), \tag{3.2}$$

where $f_V(x) := \sum_{r=0}^n \widetilde{\chi}^V(z_r) \cdot x^r$.

By Proposition 2.2, the character value $\chi^V(c^{-1})$ is typically 0, and so in order to prove Theorem 1.2 it suffices to compute f_V for only a few select choices of V. We do this now.

Proposition 3.1. *If* $V = (U, \lambda)$ *for* $U \neq \mathbf{1}$ *we have*

$$f_{U,\lambda}(x) = |G| \cdot \frac{(x; q^{-1})_n}{(q; q)_n}, \tag{3.3}$$

while if $V = (\mathbf{1}, \langle n-d, 1^d \rangle)$ *we have*

$$f_{\mathbf{1},\langle n-d,1^d\rangle}(x) = |G| \cdot \left(\frac{(x;q^{-1})_n}{(q;q)_n} + q^{-d} \cdot \sum_{m=d}^{n-1} \frac{[m]!_q \cdot [n-d-1]!_q}{[m-d]!_q \cdot [n-1]!_q} \cdot \frac{(x;q^{-1})_m}{(q;q)_m} \right). \tag{3.4}$$

The proof is a straightforward computation using Proposition 2.3, the q-binomial theorem [6, §1.3]

$$\frac{(x;q^{-1})_m}{(q;q)_m} = \frac{1}{(q;q)_m q^{\binom{m}{2}}} \sum_{k=0}^{m} (-1)^k q^{\binom{m-k}{2}} \begin{bmatrix} m \\ k \end{bmatrix}_q \cdot x^k \tag{3.5}$$

and its inverse

$$x^k = \sum_{m=0}^{k} (-1)^m q^{\binom{m}{2}} (q^k; q^{-1})_m \cdot \frac{(x;q^{-1})_m}{(q;q)_m}. \tag{3.6}$$

We continue studying the factorization generating function $F(x,y)$. We split the sum (3.2) according to whether V is a primary irreducible over the cuspidal $\mathbf{1}$ of hook shape:

$$\begin{aligned} |G| \cdot F(x,y) = & \sum_{V \neq (\mathbf{1},\langle n-d,1^d\rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) \cdot f_V(y) + \\ & \sum_{d=0}^{n-1} \deg(\mathbf{1}, \langle n-d, 1^d\rangle) \chi^{\mathbf{1},\langle n-d,1^d\rangle}(c^{-1}) \cdot f_{\mathbf{1},\langle n-d,1^d\rangle}(x) \cdot f_{\mathbf{1},\langle n-d,1^d\rangle}(y). \end{aligned} \tag{3.7}$$

We use Proposition 2.2(i) and (3.3) to rewrite the first sum on the right side of (3.7) as

$$\sum_{V \neq (\mathbf{1},\langle n-d,1^d\rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y) = |G|^2 \frac{(x;q^{-1})_n}{(q;q)_n} \frac{(y;q^{-1})_n}{(q;q)_n} \sum_{V \neq (\mathbf{1},\langle n-d,1^d\rangle)} \deg(V) \chi^V(c^{-1}).$$

Observe (following the same idea as in [10, §4.3]) that $\sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V$ is the character of the regular representation for G. It follows that $\sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) = 0$ and so one can show that

$$\sum_{V \neq (\mathbf{1},\langle n-d,1^d\rangle)} \deg(V) \chi^V(c^{-1}) \cdot f_V(x) f_V(y) = -(q;q)_{n-1} \cdot |G|^2 \cdot \frac{(x;q^{-1})_n}{(q;q)_n} \frac{(y;q^{-1})_n}{(q;q)_n}. \tag{3.8}$$

Substituting from (2.3), (3.8) and Proposition 2.2 into (3.7) yields that $F(x,y)/|G|$ equals

$$-(q;q)_{n-1} \frac{(x;q^{-1})_n}{(q;q)_n} \frac{(y;q^{-1})_n}{(q;q)_n} + \frac{1}{|G|^2} \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \cdot f_{\mathbf{1},\langle n-d,1^d\rangle}(x) \cdot f_{\mathbf{1},\langle n-d,1^d\rangle}(y). \tag{3.9}$$

In order to finish the proof of Theorem 1.2, we must extract the coefficient of $\frac{(x;q^{-1})_t}{(q;q)_t} \frac{(y;q^{-1})_u}{(q;q)_u}$ from the right side of this equation. By (3.4), this extraction reduces to a computation involving q-series. The key step is the basic hypergeometric function identity [6, (III.7)]

$${}_2\phi_1(q^{-k},\ B;\quad C;\quad z) = \frac{(C/B;q)_k}{(C;q)_k} \cdot {}_3\phi_2(q^{-k},\ B,\ Bzq^{-k}/C;\quad Bq^{1-k}/C,\ 0;\quad q).$$

3.2 Proof sketch of Theorem 1.4

We follow the same framework as in Section 3.1. Let $a_{r_1,\ldots,r_k}(q)$ be the number of tuples $(g_1,\ldots,g_k)$ of elements of $G = \mathrm{GL}_n(\mathbf{F}_q)$ such that g_i has fixed space dimension r_i for all i and $g_1 \cdots g_k = c$. Define the generating function

$$F(x_1,\ldots,x_k) = \sum_{r_1,\ldots,r_k} a_{r_1,\ldots,r_k}(q) x_1^{r_1} \cdots x_k^{r_k}.$$

The statement we wish to prove asserts a formula for this generating function when expressed in another basis. Applying Proposition 2.1, one has (as in (3.2)) that

$$|G| \cdot F(\mathbf{x}) = \sum_{V \in \mathrm{Irr}(G)} \deg(V) \chi^V(c^{-1}) f_V(x_1) \cdots f_V(x_k).$$

The same regular representation trick that leads from (3.7) to (3.9) works with more variables; it yields

$$|G| \cdot F(\mathbf{x}) = -(q;q)_{n-1} |G|^k \cdot \frac{(x_1;q^{-1})_n}{(q;q)_n} \cdots \frac{(x_k;q^{-1})_n}{(q;q)_n} + \sum_{d=0}^{n-1} (-1)^d q^{\binom{d+1}{2}} \begin{bmatrix} n-1 \\ d \end{bmatrix}_q f_{\mathbf{1},\langle n-d,1^d\rangle}(x_1) \cdots f_{\mathbf{1},\langle n-d,1^d\rangle}(x_k). \quad (3.10)$$

To finish the proof of Theorem 1.4, we must extract from this expression the coefficient of $\prod_i \frac{(x_i;q^{-1})_{p_i}}{(q;q)_{p_i}}$.

Because of the form (3.4) of the polynomial $f_{\mathbf{1},\langle n-d,1^d\rangle}(x)$, it is convenient to introduce a new parameter j, marking the number of indices p_i not equal to n. Then substituting from (3.4) into (3.10) and extracting coefficients yields the desired result.

3.3 Additional remarks on Theorem 1.4

The following remarks sketch how to recover the main theorems of [14, 10] as special cases of Theorem 1.4. Incidentally, the first remark also settles a conjecture of Lewis–Reiner–Stanton.

Remark 3.2. In [14, Thm. 1.2], Lewis–Reiner–Stanton gave a formula for the number of factorizations of a Singer cycle c as a product of ℓ reflections (that is, elements with fixed space dimension $n-1$). One can derive such a formula from Theorem 1.4 by extracting the coefficient of $x_1^{n-1} \cdots x_\ell^{n-1}$ in (1.6) applying the binomial and q-binomial theorem. It was conjectured [14, Conj. 6.3] that this formula should count factorizations of any regular elliptic element, not just a Singer cycle; since the derivation sketched here is valid for all regular elliptic elements, it settles the conjecture.

Remark 3.3. In the genus-0 case $r_1 + \ldots + r_k = (k-1)n$, there is a simple formula [10, Thm. 4.2] for $a_{r_1,\ldots,r_k}(q)$ with $0 \le r_i < n$. This formula follows from Theorem 1.4 by the triangularity of the change of basis (3.5), (3.6) and a q-analogue of the Karlsson–Minton formulas (see [6, §1.9]).

4 Application to asymptotic enumeration of factorizations by genus

In the spirit of (e.g.) [5] and [7, §4.2], one may ask to study the *asymptotic* enumeration of factorizations. Here, we compute the asymptotic growth of the number of fixed-genus factorizations of a regular elliptic element in $G = \mathrm{GL}_n(\mathbf{F}_q)$ as $n \to \infty$.

Theorem 4.1. *Let $g \geq 0$ and q be fixed. As $n \to \infty$, the number of genus-g factorizations of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ into two factors has growth rate $\Theta\left(q^{(n+g)^2/2} \,/\, |\mathrm{GL}_g(\mathbf{F}_q)|\right)$, where the implicit constants depend on q but not g.*

Remark 4.2. For the sake of comparison, we include the analogous asymptotics in $\mathfrak{S}_n$. Goupil and Schaeffer showed [7, Cor. 4.3] that for fixed $g \geq 0$, as $n \to \infty$ the number of genus-g factorizations of a fixed long cycle in $\mathfrak{S}_n$ into two factors is asymptotic to $n^{3(g-\frac{1}{2})}4^n/(g!48^g\sqrt{\pi})$.

It is interesting to observe that in both results we see that the constant depends on the size ($|\mathfrak{S}_g|$ or $|\mathrm{GL}_g(\mathbf{F}_q)|$) of a related group. Is there an explanation for this phenomenon?

The rest of this section is a sketch of the proof of Theorem 4.1. To begin, we use Theorem 1.2 to give an explicit formula for $a_{r,s}(q)$. For every positive integer g, let $P_g(x, y, z, q)$ be the following Laurent polynomial of four variables:

$$P_g(x,y,z,q) := (-1)^g q^{-g}\left(y^{-g}z^g\prod_{i=1}^{g}(yq^i-1) + y^g z^{-g}\prod_{i=1}^{g}(zq^i-1)\right) + \sum_{\substack{0\leq t',u'\leq g-1\\ 0\leq t'+u'\leq g}} (-1)^{t'+u'} \times$$
$$\begin{bmatrix} g \\ t'; u'; g-t'-u' \end{bmatrix}_q \cdot y^{u'-t'}z^{t'-u'}q^{t'u'-t'-u'}(x - yq^{t'} - zq^{u'} + 1)\prod_{i=1}^{g-t'-1}(zq^i-1)\prod_{i=1}^{g-u'-1}(yq^i-1).$$

Proposition 4.3. *If $g, r, s > 0$ satisfy $r + s = n - g$ then*

$$a_{r,s}(q) = q^{2rs+(g-1)n-\binom{g}{2}}(q^n-1)(q-1)^{-g}\cdot P_g(q^n, q^r, q^s, q) \,/\, [g]!_q.$$

Proof sketch: Extracting the coefficient of $x^r y^s$ from (1.3) using (3.5) gives

$$\frac{a_{r,s}(q)}{|G|} = (-1)^{n-g}q^{\binom{r+1}{2}+\binom{s+1}{2}} \times$$
$$\sum_{\substack{r\leq t,\ s\leq u,\\ t+u\leq n}} \frac{q^{tu-t-u-rt-su}}{(q;q)_t(q;q)_u}\begin{bmatrix} t \\ r\end{bmatrix}_q \begin{bmatrix} u \\ s\end{bmatrix}_q \frac{[n-t-1]!_q\cdot[n-u-1]!_q}{[n-1]!_q\cdot[n-t-u]!_q}\frac{(q^n-q^t-q^u+1)}{(q-1)}. \tag{4.1}$$

The rest of the proof is a long but totally unenlightening calculation: expanding the q-binomials, making the change of variables $t = r + t'$, $u = s + u'$ with $0 \leq t' + u' \leq g$, separating the $(t', u') = (g, 0)$ and $(0, g)$ terms, rearranging various factors, and doing some basic arithmetic. In particular, no nontrivial q-identities are required. □

Proof sketch of Theorem 4.1: The case $g = 0$ is straightforward from [10]. For $g > 0$, Proposition 4.3 provides an explicit polynomial formula for $a_{r,s}(q)$. Notably, the number of terms in this formula does not depend on n. It is not difficult to compute the asymptotics for the sum (over r, s such that $r + s = n - g$) of a single monomial from this polynomial; the behavior includes a constant factor that oscillates between two evaluations of the convergent Jacobi theta function $\vartheta(w,t) := \sum_{r=-\infty}^{\infty} t^{r^2}\cdot w^{2r}$. In turns out that the contribution of the single monomial $xy^{g-1}z^{g-1}$ dominates all others. To finish, one extracts the coefficient of this monomial in P_g and does some arithmetic. □

5 Closing remarks

5.1 Combinatorial proofs

Theorems 1.1 and 1.3 were originally proved in [12] by character methods (as in the outline in Section 2.2), but these are not the only known proofs. The first result (the case of two factors in $\mathfrak{S}_n$) has several combinatorial proofs, by Schaeffer–Vassilieva [18], Bernardi [1], and Chapuy–Féray–Fusy [3]. The second result (the case of k factors) has an intricate combinatorial proof [2]. It would be of interest to find combinatorial proofs of our q-analogous Theorems 1.2 and 1.4.

5.2 Other asymptotic questions

By studying the generating function in (1.3) one can show that the expected genus of a random factorization of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ into two factors is exactly

$$n - 2\sum_{t=1}^{n}(-1)^t q^{-\binom{t}{2}}(1-q^t)^{-1}. \tag{5.1}$$

Since the sum in (5.1) converges as $n \to \infty$, the vast majority of factorizations of a regular elliptic element $c \in \mathrm{GL}_n(\mathbf{F}_q)$ into two factors have large genus. Unfortunately, the techniques used to prove Theorem 4.1 are not sufficient to compute asymptotics for the number of genus-g factorizations if g grows with n. This leads to several natural questions.

Question 5.1. What is the asymptotic growth rate of the number of genus-g factorizations of a regular elliptic element in $\mathrm{GL}_n(\mathbf{F}_q)$ into two factors if g grows with n? For example, if $g = \alpha n$ for $\alpha \in (0, 1)$?

Question 5.2. Can one compute the limiting distribution of the genus of a random factorization of a regular elliptic element $c \in \mathrm{GL}_n(\mathbf{F}_q)$ into two factors when n is large? That is, choose u uniformly at random in $\mathrm{GL}_n(\mathbf{F}_q)$ and let $v = u^{-1}c$; what is the distribution of the genus of the factorization $c = u \cdot v$? As a first step, can one compute any higher moments of this distribution?

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