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On (non-) freeness of some tridendriform algebras

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Abstract. We present some results on the freeness or non freeness of some tridendriform algebras. In particular, we give a combinatorial proof of the freeness of **WQSym**, an algebra based on packed words, result already known with an algebraic proof. Then, we prove the non-freeness of an another tridendriform algebra, **PQSym**, a conjecture remained open. The method of these proofs is generalizable, in particular it has been used to prove the freeness of the dendriform algebra **FQSym** and the quadrialgebra of 2-permutations.

Résumé. Nous présentons des résultats de liberté concernant certaines algèbres tridendriformes. En particulier, nous prouvons par des arguments combinatoires que l'algèbre **WQSym** est tridendriforme libre, résultat déjà connu, mais obtenu par des méthodes purement algébriques. Puis nous prouvons que **PQSym** n'est pas une algèbre tridendriforme libre, conjecture restée ouverte jusqu'à présent. Les méthodes utilisées dans les preuves sont généralisable. En particulier, elles ont été utilisées pour prouver la liberté de l'algèbre dendriforme **FQSym** et de la quadrialgèbre des 2-permutations.

Keywords. WQSym, PQSym, tridendriform algebras, evaluation trees

1 Introduction

For some years now, a lot of algebras arise from combinatorial objects. For examples, permutations, parking functions can be equipped with dendriform products which are half shuffle products ([LR98], [NT07]). Conversely, some combinatorial objects appear naturally in the theory of operads, which is in part the study of different types of algebras and the relations between them ([LV12]). For instance, planar binary trees arise naturally from dendriform algebras ([Lod01]). Indeed, Loday proved that the Hilbert series of the free dendriform algebra over one generator is the series of Catalan numbers ([Lod01]). Other combinatorial properties have been studied in [HNT05].

Tridendriform algebras were introduced independently in [LR04] and [Cha02]. An example is given by **WQSym** ([NT06]) and the construction of this algebra comes from products defined on a family of words, the packed words. The freeness of this algebra was proved in [BR10]. Another tridendriform algebra coming from a family of words, parking functions, denoted by **PQSym** was introduced in [NT07], where the freeness of this algebra was conjectured.

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The aim of this article is to use the combinatorial interpretation of an algebra in terms of evaluation trees in order to prove the freeness of $\mathbf{WQSym}$ and to prove the non-freeness of $\mathbf{PQSym}$.

The paper is structured as follows: we present some background about evaluation trees, tridendriform and and free tridendriform algebras. Then we discuss the freeness of the algebras $\mathbf{WQSym}$ and $\mathbf{PQSym}$.

2 Background

In the sequel we are only interested in graded algebras.

2.1 Evaluation trees and algebras

Definition 1. A pair $\mathcal{A} = (A, \mathcal{P})$ is called a *$\mathcal{P}$-algebra* if A is a graded vector space ($A = \oplus_{n \in \mathbb{N}} A_n$) with A_0 isomorphic to $\mathbb{K}$, $\mathcal{P}$ is a finite set of bilinear maps from A to A such that for each B in $\mathcal{P}$, y_n in A_n and y_m in A_m, the element $B(y_n, y_m)$ is in A_{n+m}, and the element of A_0 identified to $1_{\mathbb{K}}$ is the neutral element of B. We set $A^+ := \oplus_{n \geq 1} A_n$. If $\dim(A_n)$ is finite for each n, the *Hilbert series* of A is the series $\sum_{n \geq 0} \dim(A_n) t^n$.

Definition 2. A *decorated complete binary tree* is defined by induction as follows:

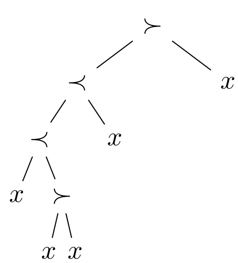
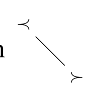
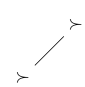
- the empty set $\emptyset$ is a decorated complete binary tree,
- the triple $(a, \emptyset, \emptyset)$ is a decorated complete binary tree, where a is an element of a certain set,
- the triple (a, T_1, T_2) is a decorated complete binary tree if T_1 and T_2 are non-empty complete binary trees, and a is an element.

We denote by $CBT(\mathcal{P}, \mathcal{B})$ the set of decorated complete binary trees where the leaves are decorated by elements of $\mathcal{B}$ and the internal nodes are decorated by the elements of $\mathcal{P}$ and by $\mathcal{BT}(\mathcal{P}, \mathcal{B})$ the graded vector space freely spanned by $CBT(\mathcal{P}, \mathcal{B})$.

Definition 3. Let $T = (a, T_1, T_2)$ and T' be two decorated binary trees. We say that T' is a *prefix* of T if T' is the empty set; or $T' = (a, T_1', T_2')$ and T_1' is a prefix of T_1 and T_2' is a prefix of T_2.

Definition 4. Let $T = (a, T_1, T_2)$ and T' be two decorated binary trees. We say that T *contains the pattern* T' if T' is a prefix of T; or T_1 or T_2 contain the pattern T'.

If T does not contain the pattern T' we say that T *avoids* the pattern T'.

Example 1. The tree  contains the pattern  but avoids  .

Definition 5. Let $(\mathcal{A}, \mathcal{P})$ be a $\mathcal{P}$-algebra, and $\mathcal{B}$ a basis of $\mathcal{A}^+$. The *vector space of evaluation trees* over $\mathcal{A}$ denoted by $\mathcal{ET}(\mathcal{A})$ is the vector space $\mathcal{BT}(\mathcal{P}, \mathcal{B})$.

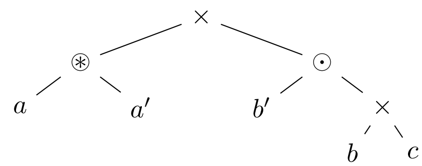


Fig. 1: Evaluation tree of $(a \circledast a') \times (b' \odot (b \times c))$.

Example 2. If $\mathcal{A} = (A, \{\times, \circledast, \odot\})$, and a, a', b, b', and c are in $\mathcal{A}$, the tree represented Figure 1 corresponds to the element $(a \circledast a') \times (b' \odot (b \times c))$ of $\mathcal{ET}(\mathcal{A})$.

Definition 6. Let $(\mathcal{A}, \mathcal{P})$ be a $\mathcal{P}$-algebra. The *evaluation* map $\mathcal{E}v$ is a linear map from $\mathcal{ET}(\mathcal{A})$ to $\mathcal{A}$ defined on trees by:

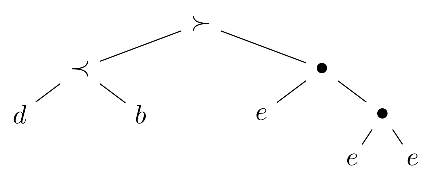
$$\begin{array}{lcl} \mathcal{E}v\,(\emptyset) & = & 1_{\mathbb{K}} \\ \mathcal{E}v\,((x,\emptyset,\emptyset)) & = & x \\ \mathcal{E}v\,((B,T_1,T_2)) & = & B\,(\mathcal{E}v(T_1), \mathcal{E}v(T_2))\,, \end{array} \tag{1}$$

where internal nodes are decorated by bilinear maps of $\mathcal{P}$ and leaves by elements of $\mathcal{A}$.

Example 3. Let us consider the vector space of non-commutative polynomials $\mathbb{K}\langle\langle A\rangle\rangle$, where A is a totally ordered alphabet. Let us define following three products on non-empty words:

$$\begin{array}{lcll} u \prec v & = & uv & \text{if } \max(u) > \max(v) \\ & = & 0 & \text{otherwise,} \\ u \succ v & = & uv & \text{if } \max(u) < \max(v) \\ & = & 0 & \text{otherwise,} \\ u \bullet v & = & uv & \text{if } \max(u) = \max(v) \\ & = & 0 & \text{otherwise.} \end{array} \tag{2}$$

For $A = \{a, b, c, \cdots, z\}$, the evaluation tree



corresponds to the expression $(d \prec b) \succ (e \bullet (e \bullet e))$ which is equal to $dbeee$.

2.2 Tridendriform algebras

Definition 7. A *tridendriform algebra* is a vector space V equipped with three bilinear maps, $\bullet$, $\succ$, $\prec$ such that for each elements a, b, c in $\mathcal{A}$, we have:

$$\begin{array}{lll} (a \prec b) \prec c = a \prec (b \odot c), & (a \succ b) \prec c = a \succ (b \prec c) & (a \odot b) \succ c = a \succ (b \succ c) \\ (a \succ b) \bullet c = a \succ (b \bullet c), & (a \prec b) \bullet c = a \bullet (b \succ c) & (a \bullet b) \prec c = a \bullet (b \prec c) \\ (a \bullet b) \bullet c = a \bullet (b \bullet c) & & \end{array} \; . \tag{3}$$

where $\odot = \prec + \bullet + \succ$.

Example 4. The non-commutative polynomials $\mathbb{K}\langle\langle A\rangle\rangle$ equipped with the three laws presented in example 3 is a tridendriform algebra ([NT06]).

2.2.1 $\mathbf{WQSym}$

One of the algebras we are interested in is the algebra $\mathbf{WQSym}$. It was defined in [NT06].

It was proved in a different way by Burgunder and Ronco in [BR10] that is a free tridendriform algebra. We give a proof based on the method presented in [Von15] for the freeness of this algebra in Section 3.

Definition 8. A *packed word* of size n is a word $w = w_1 \cdots w_n$ over the alphabet $\mathbb{N}^\star$ with the following property: if k is a letter of w, then each integer from 1 to k is also a letter of w.

We denote by $\mathcal{MT}(n)$ the set of packed word of size n, by $\mathcal{MT}$ the set of packed words, by $\mathbf{WQSym}$ the graduated vector space freely generated by packed words and by $\mathbf{WQSym}_n$ the component of degree n generated by $\mathcal{MT}(n)$.

Example 5. The word 1132422 is a packed word but 14355 is not.

Definition 9. Let w be a word over a totally ordered alphabet $\mathbb{A}$ and $b_1 < b_2 < \cdots < b_r$ the different letters of w. The word $\mathrm{pack}(w)$ is the packed word obtained by replacing all occurrences of b_i by i.

Example 6. If $w = 3944577$, we have $\mathrm{pack}(w) = 1522344$.

Definition 10. Let us consider an homogeneous basis $(\mathbf{M}_u)_{u\in\mathcal{MT}}$ of $\mathbf{WQSym}$. We define the three following products in this basis:

$$\begin{aligned} \mathbf{M}_u \prec \mathbf{M}_v &= \sum_{\substack{w=w_1w_2 \\ \mathrm{pack}(w_1)=u,\ \mathrm{pack}(w_2)=v \\ \max(w_1)>\max(w_2)}} \mathbf{M}_w \qquad \mathbf{M}_u \succ \mathbf{M}_v = \sum_{\substack{w=w_1w_2 \\ \mathrm{pack}(w_1)=u,\ \mathrm{pack}(w_2)=v \\ \max(w_1)<\max(w_2)}} \mathbf{M}_w \\ \mathbf{M}_u \bullet \mathbf{M}_v &= \sum_{\substack{w=w_1w_2 \\ \mathrm{pack}(w_1)=u,\ \mathrm{pack}(w_2)=v \\ \max(w_1)=\max(w_2)}} \mathbf{M}_w \end{aligned} \tag{4}$$

where $\max(w)$ is the greatest letter of w.

The vector space $\mathbf{WQSym}$ equipped with $\prec$, $\succ$ and $\bullet$ is a tridendriform algebra ([NT06]).

Example 7. We have:

$$\mathbf{M}_{112} \bullet \mathbf{M}_{121} = \mathbf{M}_{113232} + \mathbf{M}_{223131} + \mathbf{M}_{112121}. \tag{5}$$

2.2.2 $\mathbf{PQSym}$

In order to define the algebra $\mathbf{PQSym}$, we have to define parking functions and an algorithm the parkization ([NT07]).

Definition 11. A parking function w is a word over the alphabet $\mathbb{N}^*$ such that the non decreasing word $w\uparrow = w'_1 w'_2 \cdots w'_n$ associated to w satisfies the following property:

$$\forall i \in \{1, 2, \cdots, n\}, w_i \leq i. \tag{6}$$

Let us denote by $\mathbf{PQSym}$ the vector space freely spanned by parking functions and by $\mathbf{PQSym}_n$ the subspace spanned by parking functions of size n.

Example 8. The word $w = 11321$ is a parking function since $w\uparrow = 11123$ satisfies the property (6) but the word $v = 76191128$ is not one.

Definition 12. Let $w = w_1 \cdots w_n$ be a word over the alphabet $\mathbb{N}^*$. The *parkized* of the word denoted by $\mathbf{Park}(w)$ is obtained by the following process. We set

$$\mathbf{d}(w) = \min\{i | \sharp\{w_j \leq i\} < i\}, \tag{7}$$

by convention, the minimum of an empty set is ∞. While $\mathbf{d}(w) \leq n$, we replace w by the word obtained by replacing in w all letters j in w greater than $\mathbf{d}(w)$ by $j + \mathbf{d}(w) - a$ where a is the smallest letter of w greater than $\mathbf{d}(w)$.

Example 9. If $w = 9541843$, we obtain successively $\mathbf{d}(w) = 2$ and $w = 8431732$, then $\mathbf{d}(w) = 6$ and $w = 7431632$. Since $\mathbf{d}(w) = \infty$, we deduce that $\mathbf{Park}(w) = 7431632$.

Definition 13. Let $(\mathbf{P}_u)$ be an homogeneous basis of $\mathbf{PQSym}$. We define the following three products in this basis:

$$\mathbf{P}_u \prec \mathbf{P}_v = \sum_{\substack{w=w_1w_2 \\ \mathbf{Park}(w_1)=u,\ \mathbf{Park}(w_2)=v \\ \max(w_1)>\max(w_2)}} \mathbf{P}_w; \qquad \mathbf{P}_u \succ \mathbf{P}_v = \sum_{\substack{w=w_1w_2 \\ \mathbf{Park}(w_1)=u,\ \mathbf{Park}(w_2)=v \\ \max(w_1)<\max(w_2)}} \mathbf{P}_w$$

$$\mathbf{P}_u \bullet \mathbf{P}_v = \sum_{\substack{w=w_1w_2 \\ \mathbf{Park}(w_1)=u,\ \mathbf{Park}(w_2)=v \\ \max(w_1)=\max(w_2)}} \mathbf{P}_w \tag{8}$$

where $\max(w)$ is the greatest letter of w.

The vector space $\mathbf{PQSym}$ equipped with $\prec$, $\succ$ and $\bullet$ is a tridendriform algebra ([NT07]).

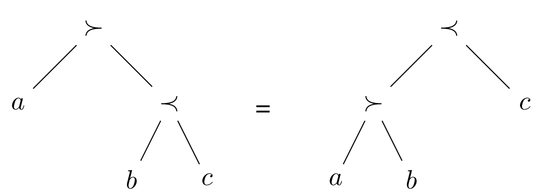
Example 10. We have :

$$\mathbf{P}_{112} \bullet \mathbf{P}_{21} = \mathbf{P}_{11221} + \mathbf{P}_{22331}. \tag{9}$$

2.2.3 Free tridendriform algebras

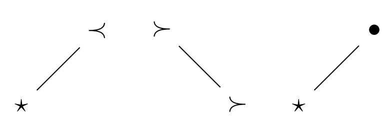
The method presented in this section has been used in [Nov14] and [Von15].

With the evaluation trees formalism, we rewrite the relations (3). For example, one of them becomes:



(10)

So the free tridendriform algebra over one generator x which we denote by $\mathcal{T}rid$ can be seen as the vector space $\mathcal{BT}(\{\prec, \succ, \bullet\}, \{x\})$ modulo the relations (3). In order to find a basis, we consider the set of trees of $CBT\ (\{\prec, \succ, \bullet\}, \{x\})$ which avoids the following patterns :



, (11)

where $\star \in \{\prec, \succ, \bullet\}$. We denote this set by $\mathbf{TAP}(\{x\})$ (for Trees Avoiding Patterns with leaves in $\{x\}$).

Proposition 1. *The generating series of* $\mathbf{TAP}(\{x\})$ *(denoted by G) is the series of the little Schroeder numbers.*

Proof. Let us denote by G_a the generating series of trees in $\mathbf{TAP}(\{x\})$ and whose root is decorated by a. So we have the following system:

$$\begin{cases} G &= t + G_{\succ} + G_{\bullet} + G_{\prec} \\ G_{\prec} &= tG \\ G_{\bullet} &= tG \\ G_{\succ} &= G(G - G_{\succ}) \end{cases} \tag{12}$$

Thus, $G_{\succ} = \frac{G^2}{1+G}$ and $G = t + 2tG + \frac{G^2}{1+G}$.

So:

$$2tG^2 + (3t-1)G + t = 0. \tag{13}$$

As a consequence, the series G satisfies the same equation as the generating series of the little Schroeder numbers. Since this equation has one and only one solution which is a series with positive integers coefficients, we deduce that the coefficients of G are the little Schroeder numbers. □

Proposition 2. *Under the relations* (3)*, each tree of* $CBT(\{\prec, \succ, \bullet\}, \{x\})$ *is a linear combination of trees in* $\mathbf{TAP}(\{x\})$.

Proof. By induction on the number of leaves n. For $n \leq 2$, it is a consequence of the rewriting rules. Let T be a tree with n+1 leaves. If T is equal to $(\succ, T_1, T_2)$. By applying the induction hypothesis on T_2, we obtain $(\succ, T_1, \circledast)$ where $\circledast$ is a linear combination of trees avoiding (11). But one of these trees may have the following shape: $(\succ, T_1, (\succ, T', T"))$, where $T"$ has a root different to $\succ$, and avoiding (11). Then we use the following rewriting rule $\succ(\succ) = (\odot)\succ$. So we have $(\succ, T_1, (\succ, T', T")) = (\succ, (\odot, T_1, T'), T")$. Thus it is sufficient to apply the induction hypothesis to the left sub-tree.

If the root is decorated by $\prec$ or $\bullet$, we first apply the induction hypothesis to the left sub-tree. If a forbidden pattern appears, at the level of the root with the left sub-tree, we apply the associated rewriting rule. Thanks to the induction hypothesis, now the left sub-tree is a leaf. If the root is now decorated by $\succ$, we already considered that case. Otherwise, it is sufficient to apply the induction hypothesis to the right sub-tree. □

Theorem 1. *A linear basis of the free tridendriform algebra over one generator x is given by the family* $\mathbf{TAP}(\{x\})$.

Proof. We have seen that family $\mathbf{TAP}(\{x\})$ spans the free tridendriform algebra over one generator x. It is known that the coefficients of the Hilbert series of this algebra is given by the little Shroeder numbers ([LR04]). Thanks to Proposition 1, we know that the generating series of $\mathbf{TAP}(\{x\})$ is also given by the generating series of the little Schroeder numbers. So the family is a basis $\mathbf{TAP}(\{x\})$. □

Theorem 2. *Let $\mathcal{A}$ be a free tridendriform algebra generated by a free family $\mathcal{F}$. A linear basis of $\mathcal{A}$ is given by the evaluations of the family* $\mathbf{TAP}(\mathcal{F})$ *(for Trees Avoiding Patterns with leaves in $\mathcal{F}$).*

Proof. Let $v \in \mathcal{A}$. Since the family $\mathcal{F}$ generates the algebra $\mathcal{A}$, the element v is a linear combination of trees in $\mathbf{TAP}\,(\mathcal{F})$. Conversely, Assume there exists a linear combination of trees in $\mathbf{TAP}\,(\mathcal{F})$ such that:

$$\sum_{T \in \mathbf{TAP}(\mathcal{F})} \alpha_T \mathcal{E}v(T) = 0. \tag{14}$$

If some of the coefficients are non-zero, we have a relation for the family $\mathcal{F}$. So each of these coefficients are zero. So $(\mathcal{E}v(T))_{T \in \mathbf{TAP}(\mathcal{F})}$ is a basis of $\mathcal{A}$. □

3 Freeness of the tridendriform algebra $\mathbf{WQSym}$

From Theorem 1, in order to prove that $\mathbf{WQSym}$ is a free tridendriform algebra, it is sufficient to find a family $\mathcal{F}$ of packed words such that the family $(\mathcal{E}v(T))_{T \in \mathbf{TAP}(\mathcal{F})}$ is a basis of $\mathbf{WQSym}$. In order to do so, we find three reduced products over packed words satisfying some relations with the three products of $\mathbf{WQSym}$ and are compatible with a total order over packed words. Thanks to the reduced products, we construct the family $\mathcal{F}$ of indecomposable packed words. Thanks to the order, we deduce that $(\mathcal{E}v(T))_{T \in \mathbf{TAP}(\mathcal{F})}$ is uni-triangular and so is a basis.

3.1 Reduced products on packed words

3.1.1 Definitions and general properties

Definition 14. Let u and v two packed words. We define the following three products:

$$u \prec' v \;=\; \bar{u}\bar{v}; \qquad u \succ' v \;=\; \hat{u}\hat{v}; \qquad u \bullet' v \;=\; \dot{u}\dot{v}\;, \tag{15}$$

where:

- $\bar{u}$ is obtained by replacing $\max(u)$ by $\max(u)+\max(v)$, $\bar{v}$ is obtained by shifting all letters of v by $\max(u)-1$,
- $\hat{u} = u$, and $\hat{v}$ is obtained by shifting all letters of v by $\max(u)$,
- $\dot{u}$ is obtained by replacing $\max(u)$ by $\max(u)+\max(v)-1$, and $\dot{v}$ is obtained by shifting all letters of v by $\max(u)-1$,

Example 11. If $u = 2134341$ and $v = 3123$, we have: $u \succ' v = 21343457567$, $u \prec' v = 21373716456$ and $u \bullet' v = 21363616456$.

Definition 15. Let u be a non-empty packed word. We say that u is *indecomposable* if :

$$u \notin \{v \star' w, \star' \in \{\prec', \succ', \bullet'\}, v \in \mathcal{MT}, w \in \mathcal{MT}, |u| \geq 1, |v| \geq 1\} \tag{16}$$

Otherwise, u is *decomposable*.

If :

$$u \notin \{v \succ' w, v \in \mathcal{MT}, w \in \mathcal{MT}, |u| \geq 1, |v| \geq 1\}, \tag{17}$$

we say that u is $\succ'$-connected.

Proposition 3. *Let u be a non-empty packed word of size n. The following two propositions are equivalent:*

1. *the word u is not $\succ'$-connected,*

2. *there exists $1 < i < n$ such that all letters $u_1 \cdots, u_i$ are strictly lesser than any letter $u_{i+1}, \cdots, u_n$.*

Proof. Assume that u is not $\succ'$-connected. There exists v and w such that:

$$u = v \succ' w = \hat{v}\hat{w}, \tag{18}$$

with the letters of $\hat{v}$ strictly lesser than any letter of $\hat{w}$. So we can take i equals to $|\hat{v}|$.

Conversely, Assume that there exists $1 < i < n$ such that all letters $u_1 \cdots, u_i$ are strictly lesser than any letter $u_{i+1}, \cdots, u_n$. Since u is a packed word, we deduce that $u = \mathrm{pack}(u_1 \cdots u_i) \succ' \mathrm{pack}(u_{i+1} \cdots u_n)$. □

Proposition 4. *Let u be a non-empty packed word of size n. The following two propositions are equivalent:*

1. *the word u is decomposable,*

2. *there exists $1 < i < n$ such that all letters $u_1 \cdots, u_i$ (but the maximum of u_i if it is one of the u_i) are strictly lesser than any letter $u_{i+1}, \cdots, u_n$.*

Proof. Assume that the word u is decomposable. So there exists v, w two packed words and $\star' \in \{\prec', \succ', \bullet'\}$ such that $u = v \star' w$. By taking $i = |v|$, we deduce that u satisfies the second property. Conversely, if u satisfies the second property, we have: $u = \mathrm{pack}(u_1 \cdots u_i) \star' \mathrm{pack}(u_{i+1} \cdots u_n)$ where $\star' = \succ'$ if any letter of $u_1, \cdots, u_i$ are strictly lesser than any letter in $u_{i+1} \cdots u_n$. If the maximum of u is in $u_1 \cdots u_i$ and in $u_{i+1} \cdots u_n$, then $u = \mathrm{pack}(u_1 \cdots u_i) \bullet' \mathrm{pack}(u_{i+1} \cdots u_n)$. If the maximum of u is only in $u_1 \cdots u_i$, then $u = \mathrm{pack}(u_1 \cdots u_i) \prec' \mathrm{pack}(u_{i+1} \cdots u_n)$. □

3.1.2 The factorization Theorem

Theorem 3. *Let w be a non-empty packed word. Then w satisfies one and only one of the following properties:*

1. *w is indecomposable;*
2. *$w = u \prec' v$ with u indecomposable;*
3. *$w = u \succ' v$ with v $\succ'$-connected;*
4. *$w = u \bullet' v$ with u indecomposable.*

In order to prove Theorem 3 we need to prove the following two propositions:

Proposition 5. *Let w be a non-empty packed word. Assume that there exists u, v, u' and v' four non-empty packed words such that:*

$$w = u \succ' v = u' \star' v' \tag{19}$$

where $\star' \in \{\prec', \bullet'\}$. Then u' is decomposable.

Proposition 6. *Let w be a non-empty packed word. Assume there exists u, v, u' and v' four non-empty packed words such that:*

$$w = u \bullet' v = u' \prec' v' \tag{20}$$

Then u' is decomposable.

Proof of Proposition 5. Assume that $u \succ' v = u' \star' v'$. By construction, we have:

$$w = \hat{u}\hat{v} = \tilde{u'}\tilde{v'}. \tag{21}$$

Since $\hat{u}$ and $\tilde{u'}$ are strict prefixes of w, one is prefix of the other. Since $\max(w)$ is not in $\hat{u}$ but in $\tilde{u'}$, it results that $\hat{u}$ is a strict prefix of $\tilde{u'}$. That is to say there exists a non-empty word ω such that :

$$\tilde{u'} = \hat{u}\omega. \tag{22}$$

But each letter of $\hat{u}$ is strictly lesser than all letters of $\hat{v}$ and so each letter of $\hat{u}$ is strictly lesser than all letters of ω. Thus, $\mathrm{pack}(\tilde{u'}) = u' = \hat{u} \succ' \omega'$. In particular, u' is decomposable. □

Proof of Proposition 6. Assume that $u \bullet' v = u' \prec' v'$. By construction, we have:

$$w = \dot{u}\dot{v} = \bar{u'}\bar{v'}. \tag{23}$$

Since $\dot{v}$ and $\bar{v'}$ are strict suffixes of w, one is suffix of the other. Since $\max(w)$ is not in $\bar{v'}$ but in $\dot{v}$, it results that $\bar{v}$ is a strict suffix of $\dot{v}$. So $\dot{u}$ is a strict prefix of $\bar{u'}$. That is to say, there exists a non-empty word ω such that

$$\bar{u'} = \dot{u}\omega. \tag{24}$$

We note that each letter of $\dot{u}$ but $\max(w)$ is strictly lesser than all letters of $\dot{v}$ and so is strictly lesser than all letters of ω. Thus, the word u' is decomposable. □

Proof of Theorem 3. Let w be a packed word of size n. Assume that w is decomposable. If w is not $\succ'$-connected, thanks to Proposition 3, there exists u and v two non-empty words such that $w = uv$, where all letters of u is strictly lesser than all letters of v. By taking u the greatest prefix satisfying this property, we deduce that $w = \mathrm{pack}(u) \succ' \mathrm{pack}(v)$ with $\mathrm{pack}(v)$ $\succ'$-connected. Assume that w is decomposable and $\succ'$-connected. Thanks to Proposition 4, there exists u and v two non-empty words such that $w = uv$, where all letters of u but $\max(u)$ is strictly lesser than all letters of v. By taking u the smallest prefix satisfying this property, we deduce that $w = \mathrm{pack}(u) \star' \mathrm{pack}(v)$ with $\mathrm{pack}(u)$ is indecomposable. If $\max(u) > \max(v)$, we have $\star' =\succ'$ and if $\max(u) = \max(v)$, we have $\star' = \bullet'$.

From Proposition 5 and Proposition 6, we deduce that the four cases are disjoint. Assume that $w = u \succ' v$ and $w = u' \succ' v'$ with v and v' $\succ'$-connected. If $|v| < |v'|$, we deduce that v' is not $\succ'$-connected. By symmetry, we deduce that $|v| = |v'|$. Then $u = u'$ and $v = v'$. By similar arguments, it follows that $u \star' v = u' \star' v'$ with $\star' \in \{\bullet', \prec'\}$ and u, u' indecomposable implies that $u = u'$ and $v = v'$.

□

Theorem 4. *There is a bijection between packed words and* $\mathbf{TAP}\ (\mathcal{F})$ *where* $\mathcal{F}$ *is the set of indecomposable packed words.*

Proof. By applying Theorem 3 recursively, we obtain the bijection. □

3.2 Freeness of $\mathbf{WQSym}$

In order to prove that $\mathbf{WQSym}$ is a free tridendriform algebra, we need to find an order on packed words which have some compatibility with the products $\prec, \succ, \bullet$ and the products $\prec', \succ', \bullet'$.

Definition 16. Let u and v be two packed words of same size. We say that u is *lesser* than v denoted by $u \leq_{m,lex} v$ if:

$$u \leq_{m,lex} v \Leftrightarrow \max(u) > \max(v) \text{ or } \max(u) = \max(v) \text{ and } u \leq_{lex} v. \tag{25}$$

Example 12. We have: $13123 \leq_{m,lex} 21122$ and $21134 \leq_{m,lex} 21143$.

Proposition 7. *Let I be a set of packed words of size m and J be a set of packed words of size n. Let u be the minimum of I and v be the minimum J. Then:*

1. $\min(I \succ J) = u \succ' v$, *2.* $\min(I \prec J) = u \prec' v$, *3.* $\min(I \bullet J) = u \bullet' v$.

Proof. Let $w = w_1 w_2$ be the minimum of $I \star J$, where $\star \in \{\prec, \succ, \bullet\}$ and w_1 be the prefix of size n. Thus w is lesser than $u \star' v$. By definition of $\leq_{m,lex}$, we deduce that $\max(w) \geq \max(u \star' v)$. By definition of w, there exists r in I and s in J such that $w \in r \star s$. Since u is the minimum of I and v in the minimum of J, we deduce that $\max(r) \leq \max(u)$ and $\max(s) \leq \max(v)$. But $\max(w) \leq \max(r) + \max(s) - 1$ if $\star = \bullet$ and $\max(w) \leq \max(r) + \max(s)$ in the other cases. It follows that $\max(r) = \max(u)$ and $\max(s) = \max(v)$. Let us denote by $\tilde{u}$ the prefix of size n of $u \star' v$ and $\tilde{v}$ the corresponding suffix. We have $w_1 \leq_{lex} \tilde{u}$. But $u \leq_{lex} \text{pack}(w_1) \leq_{lex} w_1 \leq_{lex} \tilde{u}$. But u and $\tilde{u}$ may differ only in maximum value. So $\text{pack}(w_1) = r = u$ since $\max(r)$ is equal to $\max(u)$. So we have $u \leq_{lex} w_1 \leq_{lex} \tilde{u}$ and w_1 and $\tilde{u}$ may differ only in maximum value. We have three cases:

- if $\star = \succ$, we deduce that $\tilde{u} = u$. So w_1 is equal to u.
- If $\star = \bullet$, we deduce that $\max(w) = \max(u) + \max(v) - 1$ which is also the maximum of $\tilde{u}$. So $w_1 = \tilde{u}$.
- If $\star = \prec$, we deduce that $\max(w) = \max(u) + \max(v)$ which is also the maximum of $\tilde{u}$. So $w_1 = \tilde{u}$.

In all cases, we have $w_1 = \tilde{u}$. Since $\max(w) = \max(u \star' v)$ and w_1 contains the smallest letters (but the maximum letter), we deduce that w_2 contains the other letters. So w_2 and $\tilde{v}$ have the same letters. By same arguments as previously, we deduce that $w_2 = \tilde{v}$. □

Theorem 5. *The algebra* $\mathbf{WQSym}$ *is a free tridendriform algebra.*

Proof. Thanks to Theorem 4, we deduce that the family $(\mathcal{E}v(T))_{T \in \mathbf{TAP}(\mathcal{F})}$ where $\mathcal{F}$ is the set of indecomposable packed words may be a basis. Thanks to the total order, we deduce that the matrix of the family in the basis $(\mathbf{M}_u)$ is uni-triangular. Thus the algebra $\mathbf{WQSym}$ is a free tridendriform algebra. □

4 The tridendriform algebra $\mathbf{PQSym}$

We prove that $\mathbf{PQSym}$ is not a free tridendriform algebra by expliciting relations.

Theorem 6. *The algebra* $\mathbf{PQSym}$ *is not a free tridendriform algebra.*

Proof. We denote by $\langle\mathcal{F}\rangle$ the tridendriform algebra generated by $\mathcal{F}$. Assume there exists a family $\mathcal{F}$ which generates $\mathbf{PQSym}$ as a free tridendriform algebra. Let us recall that the valuation of P in $\mathbf{PQSym}$ is the first n such that $\Pi_n(P) \neq 0$, where Π_n is the projection onto $\mathbf{PQSym}_n$.

We consider the elements in $\mathcal{F}$ which generate $\mathbf{P}_1$, $\mathbf{P}_{112}$ and $\mathbf{P}_{211}$. Since they are finite linear combinations of evaluation trees, there is a finite number of leaves. Then we can restrict to a finite free family $\mathcal{F}_n = f_1, \cdots, f_n$ elements of f which generates $\mathbf{P}_1, \mathbf{P}_{112}$ and $\mathbf{P}_{211}$. since one of the f_i is valuation one, by linear combinations we can assume without loss of generality that f_1 is the only element which has a component of degree one. Since $\Pi_2(f_1 \bullet f_1)$, $\Pi_2(f_1 \succ f_1)$ and $\Pi_2(f_1 \prec f_1)$ span the subspace $\mathbf{PQSym}_2$ we can replace f_i by

$$f_i - \alpha_i f_1 \bullet f_1 - \beta_i f_1 \succ f_1 - \gamma_i f_1 \prec f_1 \tag{26}$$

where $\alpha_i, \beta_i, \gamma_i$ are respectively the coefficients of $\mathbf{P}_{11}, \mathbf{P}_{12}, \mathbf{P}_{21}$ in f_i. From now on, we assume that for $i \geq 2$, the valuation of f_i is at least 3.

Let us denote by V the vector space spanned by $f_1, \cdots, f_n$ and by W the subspace spanned by the family of non empty evaluation trees $\mathcal{BT}(\succ, \prec, \bullet, f_1, \cdots, f_n)$.

By construction, $\mathbf{P}_1$, $\mathbf{P}_{112}$ and $\mathbf{P}_{211}$ are in the vector space $V \bigoplus W$. By hypothesis, a basis of V is given by $f_1, \cdots, f_n$. By construction, the valuation of elements in W is at least 2. So we have:

$$\mathbf{P}_1 = \alpha_1 f_1 + \sum_{i=2}^{n} \alpha_i f_i + w, \tag{27}$$

with $w \in W$ and for $i \geq 2$ the valuation of f_i is at least 3. Since the only element with valuation 1 is f_1, we deduce that $\alpha_1 \neq 0$.

Let us write $\mathbf{P}_{112}$ in the same basis. We have:

$$\mathbf{P}_{112} = \beta_1 f_1 + \sum_{k=2}^{n} \beta_i f_i + w', \tag{28}$$

with $w' \in W$. The valuation of w' is at least 2 and the valuation of $\sum_{k=2}^{n} \beta_i f_i$ is at least 3. Since the valuation of $\mathbf{P}_{112}$ is 3, we deduce that $\beta_1 = 0$. For $i \geq 2$, the valuation of f_i is at least 3. So the valuation of w' cannot be 2. Thus, the valuation of w' is at least 3. By valuation arguments, trees of size greater than 4 and trees whose leaves are different to f_1 have a valuation greater than 4. By computing the other trees, we observe that the coefficients of $\mathbf{P}_{112}$ and $\mathbf{P}_{113}$ are equal. So necessarily, the coefficient of one of the f_i for $i \neq 1$ is different to 0. Without loss of generality, by re-indexing the elements, we can assume that $\beta_2 \neq 0$.

We have:

$$\mathbf{P}_1 \bullet \mathbf{P}_{112} - \mathbf{P}_{211} \bullet \mathbf{P}_1 = \mathbf{P}_{2112} - \mathbf{P}_{2112} = 0. \tag{29}$$

And: $\mathbf{P}_1 = \sum_{i=1}^{n} \alpha_i f_i + w$, $\mathbf{P}_{112} = \sum_{k=2}^{n} \beta_i f_i + w'$, $\mathbf{P}_{211} = \sum_{i=2}^{n} \gamma_i f_i + w$".

By developing in the basis $\mathbf{TAP}$ $(\mathcal{F}_n)$, we have:

$$\sum_{1 \leq i,j \leq n} \alpha_i \beta_j f_i \bullet f_j - \sum_{1 \leq i,j \leq n} \gamma_i \alpha_j f_i \bullet f_j + \text{other terms} = 0, \tag{30}$$

where other terms are linear combination of trees of size at least three. So in fact,

$$\sum_{1\leq i,j\leq n} \alpha_i\beta_j f_i \bullet f_j - \sum_{1\leq i,j\leq n} \gamma_i\alpha_j f_i \bullet f_j = 0, \tag{31}$$

So, for $1 \leq i, j \leq n$, we have $\alpha_i\beta_j = \gamma_i\alpha_j$. Since we have $\alpha_1 \neq 0$ and $\beta_2 \neq 0$, it results that $\gamma_1 \neq 0$ and $\alpha_2 \neq 0$. So $\gamma_1\alpha_1 \neq 0$. Thus, β_1 is not equal to 0, which is a contradiction. □

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