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Fourientation activities and the Tutte polynomial: Extended abstract

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Abstract. A fourientation of a graph G is a choice for each edge of the graph whether to orient that edge in either direction, leave it unoriented, or biorient it. We may naturally view fourientations as a mixture of subgraphs and graph orientations where unoriented and bioriented edges play the role of absent and present subgraph edges, respectively. Building on work of Backman and Hopkins (2015), we show that given a linear order and a reference orientation of the edge set, one can define activities for fourientations of G which allow for a new 12 variable expansion of the Tutte polynomial T_G. Our formula specializes to both an orientation activities expansion of T_G due to Las Vergnas (1984) and a generalized activities expansion of T_G due to Gordon and Traldi (1990).

Résumé. Une fourientation d'un graphe G est un choix, pour chaque arête du graphe, d'orienter l'arête, ou laisser l'arête pas orienté, ou biorienter l'arête. Les fourientations sont un mélange d'orientations et de sous-graphes. Suivant Backman et Hopkins (2015), nous démontrons que, étant donné un ordre linéaire sur l'ensemble des arêtes et une orientation de référence, on peut définir des activités pour tous les fourientations de G qui donnent une nouvelle formule de 12 variables pour le Tutte polynôme T_G. Cette formule se spécialise à la formule de l'activité d'orientation de Las Vergnas (1984) et à la formule de l'activité généralisée de Gordon et Traldi (1990).

Keywords. activities, orientations, Tutte polynomial

1 Introduction

This is an extended abstract of a sequel paper to Backman and Hopkins (2015). In that paper, Backman and Hopkins study *fourientations* of a graph, which are a kind of generalized graph orientation. They prove that 64 *min-edge classes* of fourientations defined by restrictions on cuts and cycles are each enumerated by the Tutte polynomial of the underlying graph. The Tutte polynomial $T_G(x, y)$ of a graph G is the so-called "universal Tutte-Grothendieck invariant" (meaning that all invariants of G satisfying a deletion-contraction recurrence are a specialization of T_G) and is among the most well-studied graph polynomials; see Welsh (1999) or Welsh and Merino (2000). The connection between orientations and the Tutte polynomial goes back at least to the seminal work of Stanley (1973) who showed that the number of acyclic orientations of G is $T_G(2, 0)$. For more on the history of orientations and the Tutte polynomial, including the work of many authors who showed various classes of orientations are enumerated by the Tutte polynomial, see (Backman and Hopkins, 2015, §1.1). Backman and Hopkins were motivated specifically

by results of Gessel and Sagan (1996), as well as Hopkins and Perkinson (2016), and Backman (2014), showing that several classes of *partial orientations* are also enumerated by the Tutte polynomial. The min-edge classes defined in Backman and Hopkins (2015) put these results about partial orientations, as well as the classical Tutte polynomial evaluations for orientations, in a unified framework.

Backman and Hopkins used a direct deletion-contraction argument to prove that the Tutte polynomial enumerates min-edge classes. An alternative approach, discussed briefly in (Backman and Hopkins, 2015, §3.5), would be to define a notion of activity for fourientations. In this paper we take up that activity approach. Here "activity" refers to a certain pair of statistics of spanning trees (*internal* and *external activity*) defined by Tutte (1954). In fact, Tutte's original definition of his polynomial was in terms of internal and external activity of spanning trees. Las Vergnas (1984) defined a notion of activity for orientations and found an expansion of the Tutte polynomial in terms of orientation activities that recaptures Stanley's result as well as all other classical results about orientation classes being enumerated by the Tutte polynomial. The orientation activities formula of Las Vergnas is very analogous to another activities formula: the generalized activities formula of Gordon and Traldi (1990), which expresses the Tutte polynomial as a sum over spanning subgraphs. In this paper we offer a fourientation activities formula that simultaneously generalizes both the Las Vergnas and Gordon-Traldi formulas and in addition recovers the main result of Backman and Hopkins (2015). In this way we show how fourientations are a hybrid of (spanning) subgraphs and orientations. Indeed, fourientations appear to be a powerful tool for understanding the somewhat miraculous connection between subgraphs and orientations. This connection is also elucidated by the "active bijection" of Gioan and Las Vergnas (2005, 2009, 2015). Our approach here is basically to extend a relative of the active bijection to a $2^{|E(G)|}$-to-one surjection from fourientations to subgraphs.

2 Main results

Let G be an undirected graph which may have multiple edges and/or loops. We use $V(G)$ to denote the vertex set of G and $E(G)$ its edge set. Throughout we will use $n := \#V(G)$ for the number of vertices of G and $g := \#E(G) - \#V(G) + \kappa$ for its cyclomatic number, where $\kappa := \kappa(G)$ is the number of connected components of G. For basic background on and terminology for graphs, including such concepts as *cycles*, *cuts*, *deletion-contraction*, and the *Tutte polynomial*, see (Backman and Hopkins, 2015, §2.1) and Welsh (1999) or Welsh and Merino (2000). Recall that a *spanning subgraph* of G is a subgraph $H = (V(G), E(H))$ with $E(H) \subseteq E(G)$; i.e., it is a subgraph that includes all the vertices of G and some edges. We identify such a subgraph H with its subset of edges $S := E(H)$. Therefore we let $\mathcal{S}(G) := 2^{E(G)}$ denote the set of spanning subgraphs of G. For C an undirected cut or cycle of G we use $E(C)$ to denote the set of edges of C. Let $S \in \mathcal{S}(G)$. Abusing notation, we say that a cut Cu of G is a *cut of* S if $E(\mathrm{Cu}) \cap S = \emptyset$. Similarly we say that a cycle Cy of G is a *cycle of* S if $E(\mathrm{Cy}) \subseteq S$.

Let $<$ be a total order on $E(G)$. Gordon and Traldi (1990) define cut and cycle activities for arbitrary spanning subgraphs of G that generalize Tutte's original definition of activity for spanning trees in Tutte (1954). Specifically, we say $e \in E(G)$ is *cut active* in S if it is the min edge (where "min" is the minimum according to $<$) in $E(\mathrm{Cu})$ for Cu some cut of $S \setminus \{e\}$. We use $I(S)$ (where I is for *isthmus*) to denote the cut active edges of S. Similarly, we say that e is *cycle active* in S if it is the min edge in $E(\mathrm{Cy})$ for Cy some cycle of $S \cup \{e\}$. We use $L(S)$ (where L is for *loop*) to denote the cycle active edges of S. Observe that the maps $I, L \colon \mathcal{S}(G) \to \mathcal{S}(G)$ depend on the edge order $<$ but we leave this dependence implicit. One easy consequence of the definitions is that $\hat{I}(S) \cap \hat{L}(S) = \varnothing$ for all $S \in \mathcal{S}(G)$. We now review a few other basic results about the generalized activities $\hat{I}$ and $\hat{L}$.

Lemma 2.1 *Let $S, T \in \mathcal{S}(G)$ with $S \setminus (\hat{I}(S) \cup \hat{L}(S)) \subseteq T \subseteq S \cup \hat{I}(S) \cup \hat{L}(S)$. Then $\hat{I}(S) = \hat{I}(T)$ and $\hat{L}(S) = \hat{L}(T)$.*

Lemma 2.2 *We have $\left|\hat{I}(S) \setminus S\right| = \kappa(S) - \kappa$ and $\left|\hat{L}(S) \cap S\right| = \kappa(S) + |S| - n$ for all $S \in \mathcal{S}(G)$.*

The main result about these subgraph activities is (Gordon and Traldi, 1990, Theorem 3), which provides the following expansion of the Tutte polynomial $T_G(x, y)$ of G.

Theorem 2.3 ((Gordon and Traldi, 1990, Theorem 3); see also (Las Vergnas, 2013, Theorem 3.5)) *For any graph G and any total order $<$ on $E(G)$ we have*

$$T_G(x_* + w_*, y_* + z_*) = \sum_{S \in \mathcal{S}(G)} {x_*}^{\#I(S)\cap S} {w_*}^{\#I(S)\setminus S} {y_*}^{\#L(S)\setminus S} {z_*}^{\#L(S)\cap S}.$$

Theorem 2.3 gives a very general expansion that recovers many other well-known expansions of the Tutte polynomial. For instance, taking $w_* := 0$ and $z_* := 0$ recovers Tutte's spanning tree activity expansion for T_G; see (Tutte, 1954, (13)):

$$T_G(x_*, y_*) = \sum_{T \text{ spanning tree of } G} {x_*}^{\#I(T)} {y_*}^{\#L(T)}.$$

And taking $x_* := 1$ and $y_* := 1$ recovers the rank generating function expansion for the Tutte polynomial; see (Welsh, 1999, (2.5)) or (Welsh and Merino, 2000, (8)):

$$T_G(1 + w_*, 1 + z_*) = \sum_{S \in \mathcal{S}(G)} {w_*}^{\kappa - \kappa(S)} {z_*}^{\kappa + \#S - n}.$$

Here $\kappa(S) := \kappa(V(G), S)$ is the number of connected components of the spanning subgraph $(V(G), S)$. Moreover, from Theorem 2.3 it follows that the number of spanning forests of G is $T_G(2, 1)$, and, for G connected, the number of connected spanning subgraphs is $T_G(1, 2)$ and the number of spanning trees is $T_G(1, 1)$. In fact, by taking $x_*, y_*, w_*, z_* \in \{0, 1\}$ this expansion gives combinatorial interpretations for all evaluations $T_G(a, b)$ with integer $0 \leq a, b \leq 2$ in terms of spanning subgraphs and activity.

There is a very analogous story for orientation activities due to Las Vergnas (1984).[(i)] In order to talk about orientations of G it is helpful to have a fixed reference orientation O_{ref}. The reference orientation O_{ref} is a choice for each edge $e = \{u, v\} \in E(G)$ of a positive direction $e^+ = (u, v)$ and therefore also a negative direction $e^- = (v, u)$. This can be seen as a choice of oriented matroid whose underlying matroid is the cycle matroid of G. With respect to O_{ref} an orientation of G is then just a subset $O \subseteq \mathbb{E}(G)$ of the set $\mathbb{E}(G) := \{e^+, e^- \colon e \in E(G)\}$ which satisfies $\#\{e^+, e^-\} \cap O = 1$ for all $e \in E(G)$. Here we identify an orientation O (which we have defined to be just a set of formal symbols) with the set of directed edges $\{(u, v) \colon e^\delta = (u, v) \in O, \delta \in \{+, -\}\}$ and this identification depends implicitly on O_{ref}. We use $\mathcal{O}(G)$ to denote the set of orientations of G. Note that $\#\mathcal{O}(G) = 2^{\#E(G)}$ even when G has loops or multiple edges. For $\overrightarrow{C}$ a directed cut or cycle of G we use $\mathbb{E}(\overrightarrow{C})$ to denote the set of directed edges of $\overrightarrow{C}$ thought of as a subset of $\mathbb{E}(G)$, and $E(\overrightarrow{C})$ its set of undirected edges. Let $O \in \mathcal{O}(G)$. We say that a

[(i)] Orientation activities were first introduced by Berman (1977), but his account of their connection with the Tutte polynomial was imprecise. See the footnote on page 370 of Las Vergnas (1984) for details.

directed cut (cycle) $\overrightarrow{C}$ of G is a *directed cut (cycle) of* O if $\mathbb{E}(\overrightarrow{C}) \subseteq O$. In other words, a directed cut (cycle) of an orientation of a graph is a cut (cycle) of the graph where all the edges are oriented consistently in the orientation.

Las Vergnas defines his cycle activities as follows. Again we must fix a total order $<$ of $E(G)$. We say that $e \in E(G)$ is *cut (cycle) active* in the orientation O if it is the minimum edge in $E(\overrightarrow{C})$ for $\overrightarrow{C}$ some directed cut (cycle) of O. We let $I(O)$ denote the set of cut active edges of O and $L(O)$ the set of cycle active edges. In order to state the orientation analog of Theorem 2.3 we need one more piece of notation. For $O \in \mathcal{O}(G)$ and $\delta \in \{+,-\}$ set $O^\delta := \{e \in E(G) \colon e^\delta \in O\}$ so that $E(G) = O^+ \sqcup O^-$. To simplify notation we write $I(O^+) := I(O) \cap O^+$ and so on.

Theorem 2.4 ((Las Vergnas, 1984, Theorem 3.1) and (Las Vergnas, 2012, Theorem 3.1))
For any graph G, any reference orientation O_{ref}, and any total order $<$ on $E(G)$ we have

$$T_G(x+w, y+z) = \sum_{O \in \mathcal{O}(G)} x^{\#I(O^+)} w^{\#I(O^-)} y^{\#L(O^+)} z^{\#L(O^-)}.$$

Again, Theorem 2.4 is a very general (and elegant) expansion for the Tutte polynomial with many consequences. For instance, taking $x := w := 1$, and $y := z := 0$ we recover the celebrated result that the number of acyclic orientations of G is $T_G(2,0)$ from Stanley (1973). Dually, taking $x := w := 0$, and $y := z := 1$ yields Las Vergnas's own result that the number of strongly connected orientations of G is $T_G(0,2)$, which he proved in Las Vergnas (1980). And more generally by taking $x, y, w, z \in \{0,1\}$ this expansion gives combinatorial interpretations for all $T_G(a,b)$ with $0 \leq a, b \leq 2$ in terms of orientations. This 3×3 square of orientation classes has been explored in the unifying works of Gioan (2007) (see also Gioan (2008)) and Bernardi (2008).

Of course, when the variables with asterisks are set equal to those without, Theorems 2.3 and 2.4 offer two different combinatorial expansions for the same polynomial. Proving bijectively that these expressions are indeed equal by matching terms on either side is one aim of the so-called "active bijection" of Gioan and Las Vergnas (2009). Here we connect Theorems 2.3 and 2.4 in a different way: we offer an expansion of the Tutte polynomial in terms of fourientation activities that simultaneously generalizes both the Gordon-Traldi and Las Vergnas formulas. These "fourientations" can therefore be seen as a hybrid of subgraphs and orientations.

Recall from Backman and Hopkins (2015) that a *fourientation* of a graph G with respect to some fixed reference orientation O_{ref} is just an arbitrary subset of $\mathbb{E}(G)$. We use $\mathcal{O}^4(G)$ to denote the set of fourientations of G. Let $O \in \mathcal{O}^4(G)$. We say that $e \in E(G)$ is *unoriented* in O if $\{e^+, e^-\} \cap O = \emptyset$ and we say that e is *bioriented* in O if $\{e^+, e^-\} \subseteq O$. We say e is *simply oriented*, or just *oriented*, in O if it is neither unoriented nor bioriented. Let $O^o \in \mathcal{S}(G)$ denote the set of oriented edges of O, $O^u \in \mathcal{S}(G)$ the set of unoriented edges, and $O^b \in \mathcal{S}(G)$ the set of bioriented edges. Let O^+ denote the set of oriented edges of O oriented in agreement with O_{ref} and O^- the set of oriented edges oriented in disagreement with O_{ref} so that $O^o = O^+ \sqcup O^-$.

The key definition in Backman and Hopkins (2015) is that of a potential cut (cycle) of a fourientation, which we now review. Let $O \in \mathcal{O}^4(G)$ be a fourientation. We say a directed cut $\overrightarrow{\mathrm{Cu}}$ of G is a *potential cut of* O if $e^\delta \in \mathbb{E}(\overrightarrow{\mathrm{Cu}}) \Rightarrow e^{-\delta} \notin O$. Similarly, a directed cycle $\overrightarrow{\mathrm{Cy}}$ of G is a *potential cycle of* O if $e^\delta \in \mathbb{E}(\overrightarrow{\mathrm{Cy}}) \Rightarrow e^\delta \in O$. In other words, a potential cut of a fourientation is the same as a directed cut of an orientation except that some edges of the cut are allowed to be unoriented; and a potential cycle of

a fourientation is the same as a directed cycle of an orientation expect that some edges of the cycle are allowed to be bioriented. Backman and Hopkins (2015) enumerate many classes of fourientations defined in terms of potential cuts and cycles.

In what follows we will define some fourientation activities $I, L\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$. For convenience we use the notation $I^o(O) := I(O) \cap O^o$, $L^b(O) := L(O) \cap O^b$ and so on. Some of these sets are defined individually:

Definition 2.5 Let G be a graph, O_{ref} a reference orientation, and $<$ a total order on $E(G)$. Also let $\sigma_u, \sigma_b\colon E(G) \to \{+,-\}$ be arbitrary sign labels of the edges of G. Then for each $O \in \mathcal{O}^4(G)$, we set

1. $I^o(O) := \{e \in O^o\colon e \text{ is the min edge in some potential cut of } O\}$;
2. $L^o(O) := \{e \in O^o\colon e \text{ is the min edge in some potential cycle of } O\}$;
3. $I^u(O) := \{e \in O^u\colon e \text{ is the min edge in some potential cut of } O \cup \{e^{\sigma_u(e)}\}\}$;
4. $L^b(O) := \{e \in O^b\colon e \text{ is the min edge in some potential cycle of } O \setminus \{e^{\sigma_b(e)}\}\}$.

From now on fix G, O_{ref}, $<$, σ_u, and σ_b as in Definition 2.5. The following is our main result.

Theorem 2.6 *There are fourientation cut and cycle activities $I, L\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$, as well as a map $\varphi\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$, such that*

$$(k_1+m)^{n-\kappa}(k_2+l)^g T_G\left(\frac{k_1x+k_2w+m\hat{x}+l\hat{w}}{k_1+m}, \frac{k_2y+k_1z+l\hat{y}+m\hat{z}}{k_2+l}\right) =$$
$$\sum_{O\in\mathcal{O}^4(G)} k_1^{|O^o\cap\varphi(O)|}k_2^{|O^o\setminus\varphi(O)|}l^{|O^u|}m^{|O^b|}x^{|I^+(O)|}w^{|I^-(O)|}\hat{x}^{|I^b(O)|}\hat{w}^{|I^u(O)|}y^{|L^+(O)|}z^{|L^-(O)|}\hat{y}^{|L^u(O)|}\hat{z}^{|L^b(O)|}.$$

In addition:

- *the mappings I, L are compatible with Definition 2.5 in the sense that for every $O \in \mathcal{O}^4(G)$, $I(O) \cap O^o = I^o(O)$, $I(O) \cap O^u = I^u(O)$, $L(O) \cap O^o = L^o(O)$ and $L(O) \cap O^b = L^b(O)$;*
- *the mappings I, L are also compatible with the Gordon-Traldi generalized activities, in the sense that if $O^o = \varnothing$ then $I(O) = \hat{I}(O^b)$ and $L(O) = \hat{L}(O^b)$.*

There is of course a natural embedding $\mathcal{O}(G) \hookrightarrow \mathcal{O}^4(G)$ (the identity map), but there is also a natural embedding $\mathcal{S}(G) \hookrightarrow \mathcal{O}^4(G)$ whereby $S \mapsto O_S$, with e unoriented in O_S if $e \notin S$ and e bioriented in O_S if $e \in S$. In this way, Theorem 2.6 recovers Theorem 2.3 by taking $(k,l,m) := (0,1,1)$ and recovers Theorem 2.4 by taking $(k,l,m) := (1,0,0)$.

Moreover, Theorem 2.6 offers a nontrivial interpolation of the Gordon-Traldi and Las Vergnas formulas. For instance, taking $(k,l,m) := (1,1,0)$ and $y := z := 0$ and $x := w := x_* := w_* := y_* := z_* := 1$, the formula says that the number of acyclic partial orientations of G is $2^g T_G(3,1/2)$, a result originally obtained by Gessel and Sagan (1996).[ii] Furthermore, (Backman and Hopkins, 2015, §4.4) shows that, when G is connected, for any choice of root $q \in V(G)$ there is a choice of reference orientation O_{ref}, total edge order $<$, and edge labels σ_u, σ_b so that the q-connected fourientations of G are

[ii] See Backman and Hopkins (2015) for precise definitions of *partial orientation*, *acyclic partial orientation*, *q-connected partial orientation*, *min-edge class*, et cetera.

precisely those fourientations $O \in \mathcal{O}^4(G)$ which satisfy $I(O^u) = I(O^-) = \emptyset$ (where I is as in Theorem 2.6). Thus Theorem 2.6 also implies that (for connected graphs G) the number of q-connected fourientations is $2^{\#E(G)}T_G(1,2)$ and the number of acyclic, q-connected partial orientations is $2^g T_G(1,1/2)$, two other evaluations first obtained by Gessel and Sagan (1996). More generally, taking $x_* := y_* := 1$ and $x, y, w, z, w_*, z_* \in \{0,1\}$ we recover all the enumerations of (in fact, generating functions for) "min-edge classes" of fourientations obtained by Backman-Hopkins. (These min-edge classes form an intersection lattice of 64 sets of fourientation defined in terms of restrictions on potential cuts and cycles.) So Theorem 2.6 encompasses the work of Backman-Hopkins as well as that of Gordon-Traldi and Las Vergnas.

In fact, Theorem 2.6 follows easily from the following theorem:

Theorem 2.7 *There exists a $2^{|E(G)|}$-to-one surjection $\varphi\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$ such that*

- *the map $O \mapsto (\varphi(O), O^o)$ is a bijection between $\mathcal{O}^4(G)$ and $\mathcal{S}(G) \times \mathcal{S}(G)$;*
- *we have $O^u \cap \varphi(O) = \varnothing$ and $O^b \subseteq \varphi(O)$ for all $O \in \mathcal{O}^4(G)$;*
- *setting $I(O) := \hat{I}(\varphi(O))$ and $L(O) := \hat{L}(\varphi(O))$ (where these are the Gordon-Traldi generalized activities with respect to the same edge order $<$), the resulting maps $I, L\colon \mathcal{O}^4(G) \to \mathcal{S}(G)$ satisfy the compatibility conditions in Theorem 2.6;*
- *with $I(O)$ and $L(O)$ as in the previous bullet point, we have $I^+(O) \subseteq \varphi(O)$, $I^-(O) \cap \varphi(O) = \varnothing$, $L^-(O) \subseteq \varphi(O)$, and $L^+(O) \cap \varphi(O) = \varnothing$ for all $O \in \mathcal{O}^4(G)$.*

Proof of Theorem 2.6 from Theorem 2.7: By applying Theorem 2.3, we have

$$\begin{aligned}
&(k_1+m)^{n-\kappa}(k_2+l)^g T_G\left(\frac{k_1x+k_2w+m\hat{x}+l\hat{w}}{k_1+m}, \frac{k_2y+k_1z+l\hat{y}+m\hat{z}}{k_1+l}\right) = \\
&(k_1+m)^{n-\kappa}(k_2+l)^g \sum_{S\in\mathcal{S}(G)} \left\{ \begin{array}{c} \left(\frac{k_1x+m\hat{x}}{k_1+m}\right)^{|\hat{I}(S)\cap S|} \left(\frac{k_2w+l\hat{w}}{k_1+m}\right)^{|\hat{I}(S)\setminus S|} \\ \cdot \left(\frac{k_2y+l\hat{y}}{k_2+l}\right)^{|\hat{L}(S)\setminus S|} \left(\frac{k_1z+m\hat{z}}{k_2+l}\right)^{|\hat{L}(S)\cap S|} \end{array} \right. = \\
&\sum_{S\in\mathcal{S}(G)} \left\{ \begin{array}{c} (k_1+m)^{n-\kappa-|\hat{I}(S)|}(k_2+l)^{g-|\hat{L}(S)|} \\ \cdot (k_1x+m\hat{x})^{|\hat{I}(S)\cap S|}(k_2w+l\hat{w})^{|\hat{I}(S)\setminus S|}(k_2y+l\hat{y})^{|\hat{L}(S)\setminus S|}(k_1z+m\hat{z})^{|\hat{L}(S)\cap S|} \end{array} \right. = \\
&\sum_{S\in\mathcal{S}(G)} \left\{ \begin{array}{c} (k_1+m)^{|S\setminus(\hat{I}(S)\cup\hat{L}(S))|}(k_2+l)^{|E(G)\setminus(S\cup\hat{I}(S)\cup\hat{L}(S))|} \\ \cdot (k_1x+m\hat{x})^{|\hat{I}(S)\cap S|}(k_2w+l\hat{w})^{|\hat{I}(S)\setminus S|}(k_2y+l\hat{y})^{|\hat{L}(S)\setminus S|}(k_1z+m\hat{z})^{|\hat{L}(S)\cap S|} \end{array} \right. = \\
&\sum_{S\in\mathcal{S}(G)} \sum_{O\in\varphi^{-1}(S)} \left\{ \begin{array}{c} k_1^{|O^o\cap\varphi(O))|}k_2^{|O^o\setminus\varphi(O)|}l^{|O^u|}m^{|O^b|} \\ \cdot x^{|I^+(O)|}w^{|I^-(O)|}\hat{x}^{|I^b(O)|}\hat{w}^{|I^u(O)|}y^{|L^+(O)|}z^{|L^-(O)|}\hat{y}^{|L^u(O)|}\hat{z}^{|L^b(O)|} \end{array} \right. = \\
&\sum_{O\in\mathcal{O}^4(G)} k_1^{|O^o\cap\varphi(O)|}k_2^{|O^o\setminus\varphi(O)|}l^{|O^u|}m^{|O^b|}x^{|I^+(O)|}w^{|I^-(O)|}\hat{x}^{|I^b(O)|}\hat{w}^{|I^u(O)|}y^{|L^+(O)|}z^{|L^-(O)|}\hat{y}^{|L^u(O)|}\hat{z}^{|L^b(O)|}.
\end{aligned}$$

The fourth line above follows from the third because for all $S \in \mathcal{S}(G)$ we have

$$\begin{aligned}
n - \kappa - \left|\hat{I}(S)\right| &= \left|S \setminus (\hat{I}(S) \cup \hat{L}(S))\right|; \\
g - \left|\hat{L}(S)\right| &= \left|E(G) \setminus (S \cup \hat{I}(S) \cup \hat{L}(S))\right|
\end{aligned}$$

as a consequence of Lemma 2.2. And the fifth line above follows from the fourth because of the conditions of Theorem 2.7 (which for example force $I^+(O) \cup I^b(O) = \hat{I}(S) \cap S$ for $S \in \mathcal{S}(G)$, $O \in \varphi^{-1}(S)$). $\square$

So most of the work presented in Backman, Hopkins, and Traldi (2015) is directed toward proving Theorem 2.7, by defining the map φ and verifying its properties. Note in particular that φ will be a left inverse to the embedding $\mathcal{S}(G) \hookrightarrow \mathcal{O}^4(G)$, and φ will restrict to a bijection $\mathcal{O}(G) \xrightarrow{\sim} \mathcal{S}(G)$ that will turn out to be an example of the activity-preserving bijections of Gioan-Las Vergnas.

We define φ recursively by deleting and contracting the maximum edge $e_{\max}$ of G (according to $<$). For $O \in \mathcal{O}^4(G)$ and $e \in E(G)$, we use the notation $O - e$ to denote the fourientation of $G - e$ obtained from O by restricting to $\mathbb{E}(G - e)$. The notation O/e is used similarly (in fact, as sets $O - e = O/e$). In the generic case where $e_{\max}$ is neither an isthmus nor a loop, we depend on the following key lemma (which should be compared to (Las Vergnas, 1984, Lemma 3.2) and (Las Vergnas, 2012, Lemma 3.4)):

Lemma 2.8 *For $e \in E(G)$ and $O \in \mathcal{O}^4(G)$ set ${}^eO := O\Delta\{e^+, e^-\}$ where Δ denotes set-theoretic symmetric difference. Suppose the maximum edge $e := e_{\max}$ of G is neither an isthmus nor a loop. Then at least one of the following holds:*

1. • $I(O^o) = I((O/e)^o)$, $I(O^u) = I((O/e)^u)$, $L(O^o) = L((O/e)^o)$, $L(O^b) = L((O/e)^b)$,
   • $I({}^eO^o) = I((O-e)^o)$, $I({}^eO^u) = I((O-e)^u)$, $L({}^eO^o) = L((O-e)^o)$, $L({}^eO^b) = L((O-e)^b)$;
2. • $I({}^eO^o) = I((O/e)^o)$, $I({}^eO^u) = I((O/e)^u)$, $L({}^eO^o) = L((O/e)^o)$, $L({}^eO^b) = L((O/e)^b)$,
   • $I(O^o) = I((O-e)^o)$, $I(O^u) = I((O-e)^u)$, $L(O^o) = L((O-e)^o)$, $L(O^b) = L((O-e)^b)$.

(Note that (2) merely asserts that (1) holds for ${}^eO$.) Moreover, if e is bioriented in O then certainly (1) holds and if e is unoriented in O then certainly (2) holds.

In the case where $e_{\max}$ is either an isthmus or a loop we apply one the following two simpler lemmas:

Lemma 2.9 *Suppose the maximum edge $e := e_{\max}$ of G is an isthmus. Then*

- $I(O^o) \setminus \{e\} = I((O-e)^o)$, $I(O^u) \setminus \{e\} = I((O-e)^u)$;
- $L(O^o) = L((O-e)^o)$, $L(O^b) = L((O-e)^b)$.

Lemma 2.10 *Suppose the maximum edge $e := e_{\max}$ of G is a loop. Then*

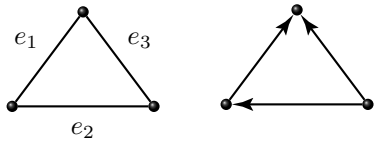
- $I(O^o) = I((O/e)^o)$, $I(O^u) = I((O/e)^u)$;
- $L(O^o) \setminus \{e\} = L((O/e)^o)$, $L(O^b) \setminus \{e\} = L((O/e)^b)$.

Given the above lemmas it is straightforward to prove Theorem 2.7 by recursively defining φ. When $e_{\max}$ is unoriented we send it to an absent edge and when it is bioriented we send it to a present edge; what happens when $e_{\max}$ is oriented depends on which condition of Lemma 2.8 holds. Lemmas 2.9, and 2.10 are used to deal with the "base cases" of isthmuses and loops.

As mentioned, Lemmas 2.9 and 2.10 are straightforward. Thus the bulk of the proof of Theorem 2.7 lies in verifying the key lemma, Lemma 2.8, which we will not do in this extended abstract because of length considerations. The proof of Lemma 2.8 is rather technical and long (it takes up a 20 page appendix in Backman, Hopkins, and Traldi (2015)) but not especially difficult. Let us conclude this extended abstract by giving an example of the map φ and making some more remarks.

3 Example and remarks

In this section we will give an example of what the surjection φ is in one of the simplest nontrivial cases: the triangle graph. So let G and O_{ref} be as below:

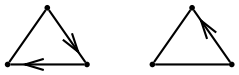


G $\qquad$ O_{ref}

We take the total edge order $e_1 < e_2 < e_3$. We set $\sigma_u(e) := -$ and $\sigma_b(e) := +$ for all edges $e \in E(G)$. Then Figures 1 and 2 show the fibers $\varphi^{-1}(S)$ for all $S \in \mathcal{S}(G)$. In our depiction of a subgraph $S \in \mathcal{S}(G)$ edges that belong to S are solid and edges that do not belong to S are dashed. And in our depiction of a fourientation $O \in \mathcal{O}^4(G)$ oriented edges $e^\delta = (u, v)$ are depicted by an arrow from u to v, while unoriented edges are solid lines with no arrows, and bioriented edges are drawn with two arrows, one in each direction.

Now let us make a few remarks about how the map φ can be rather subtle and also comment on some possible future directions:

1. For some $S \in \mathcal{S}(G)$, there can be $O_1, O_2 \in \varphi^{-1}(S)$ with $O_1^+ \cap O_2^- \neq \emptyset$. For instance, in the example above, for the following fourientations:



O_1 $\qquad$ O_2

we have $O_1, O_2 \in \varphi^{-1}(\{e_3\})$ but $e_3 \in O_1^-$ while $e_3 \in O_2^+$. Thus φ is not just a direct interpolation between S and the one simple orientation in $\varphi^{-1}(S)$. Put differently, it does not seem possible to deduce Theorem 2.6 from Theorems 2.3 and 2.4 alone.

2. Explicitly describing $I(O^b)$ and $L(O^u)$ using the intrinsic properties of a fourientation O remains an interesting open problem. One particularly intriguing fact is that whether an edge $e \in E(G)$ belongs to $I(O^b)$ depends on more than just the status in O of all edges $f \in E(G)$ with $f \geq e$. For instance, in the example above, for the following fourientations we have $e_2 \in I(O_1^b)$ but $e_2 \notin I(O_2^b)$ even though O_1 and O_2 look the same when restricted to $\{e_2, e_3\}$:

O_1 $\qquad$ O_2

This is in marked contrast to the situation for $I(O^o)$, $L(O^o)$, $I(O^u)$, $L(O^b)$, or indeed the Gordon-Traldi or Las Vergnas activities where an edge's being active depends only on the status of edges greater than or equal to it in the total edge order. Somehow $I(O^u)$ and $L(O^b)$ are "local" (see the versatility of σ_u and σ_b) whereas $I(O^b)$ and $L(O^u)$ must be "global."

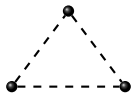
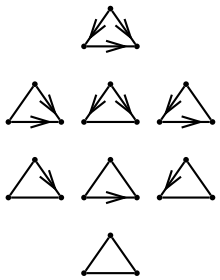
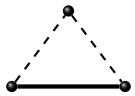
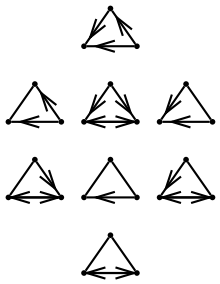
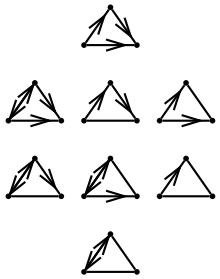
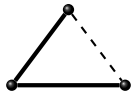
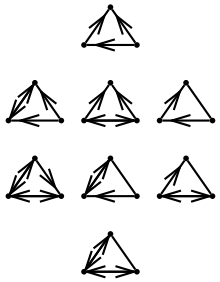
$S \in \mathcal{S}(G)$	$I(S)$	$L(S)$	$\varphi^{-1}(S)$
	$\{e_1, e_2\}$	$\emptyset$	
	$\{e_1, e_2\}$	$\emptyset$	
	$\{e_1, e_2\}$	$\emptyset$	
	$\{e_1, e_2\}$	$\emptyset$	

Fig. 1: The first half of the $S \in \mathcal{S}(G)$ for the example in §3, together with their activities and fibers $\varphi^{-1}(S)$.

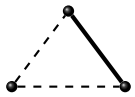
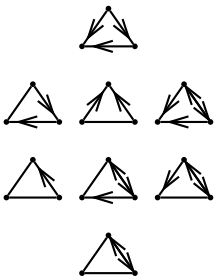
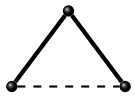
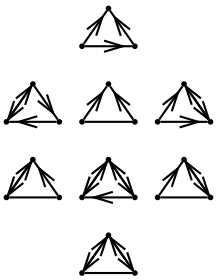
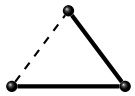
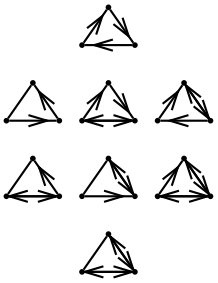
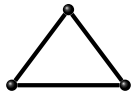
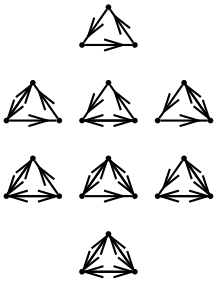
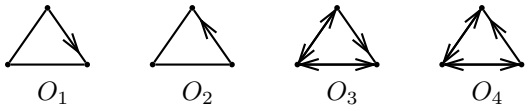
$S \in \mathcal{S}(G)$	$I(S)$	$L(S)$	$\varphi^{-1}(S)$
	$\{e_1\}$	$\emptyset$	
	$\{e_1\}$	$\emptyset$	
	$\emptyset$	$\{e_1\}$	
	$\emptyset$	$\{e_1\}$	

Fig. 2: The second half of the $S \in \mathcal{S}(G)$ for the example in §3, together with their activities and fibers $\varphi^{-1}(S)$.

3. Another instance of the complicated nature of fourientation activities is that fourientations which seem to be related in a natural way may have different activities.



   For instance, the four pictured fourientations are related through orientation reversals, but they exhibit all three different activity patterns that occur in K_3.

4. In (Backman and Hopkins, 2015, §4), Backman and Hopkins show how min-edge classes of fourientations (which as mentioned earlier are enumerated by Theorem 2.6) appear in various algebraic settings. It would be extremely interesting to try to interpret the equation in Theorem 2.6 algebraically, e.g., as a formula for the Betti numbers of some polyhedral complex or some polynomial ideal.

5. Theorem 2.6 could be extended to reorientations of an orientable matroid, but we restrict our discussion to graphs for simplicity and clarity. Indeed, part of what made the work of Las Vergnas, especially the paper Las Vergnas (1984), opaque to us on first reading was the level of generality at which he was working. Las Vergnas actually works at the level of *matroid perspectives*, a step above matroids.

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