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A noncommutative geometric LR rule

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Abstract The geometric Littlewood-Richardson (LR) rule is a combinatorial algorithm for computing LR coefficients derived from degenerating the Richardson variety into a union of Schubert varieties in the Grassmannian. Such rules were first given by Vakil and later generalized by Coskun. In this paper we give a noncommutative version of the geometric LR rule. As a consequence, we establish a geometric explanation for the positivity of noncommutative LR coefficients in certain cases.

Résumé La règle de Littlewood–Richardson (LR) géométrique est un algorithme combinatoire de calcul des coefficients de LR, conçu à partir de l'étude des dégénérations d'une variété de Richardson en variétés de Schubert, dans la Grassmannienne. Vakil a été le premier à donner une telle règle, et Coskun en a produit des généralisations. Dans cet article, nous donnons une version non–commutative de la règle de LR géométrique. Comme conséquence, nous obtenons une explication géométrique de la positivité de certains coefficients de LR non-commutatifs.

Keywords. Grassmannians, Mondrian tableaux, noncommutative symmetric functions, Schubert calculus, symmetric functions

1 Introduction

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-dimensional vector spaces in $\mathbb{C}^n$. For any partition λ whose Young diagram is contained in a $k \times (n-k)$ rectangle, let σ_λ denote the corresponding Schubert class in the cohomology ring $H^*(\mathrm{Gr}(k,n))$. The Schubert classes $\{\sigma_\lambda\}$ form a basis of $H^*(\mathrm{Gr}(k,n))$ and we have the following cup product expansion

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda\mu}\sigma_\nu$$

where the sum is over all partitions ν whose Young diagram is contained in a $k \times (n-k)$ rectangle. The $c^\nu_{\lambda\mu}$ are called Littlewood-Richardson (LR) coefficients and are nonnegative integers. That they also arise in the following setting is one of the most important and rich source of mathematics in the last century.

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For a partition λ, let s_λ denote the corresponding Schur function in the algebra of symmetric functions Sym. The Schur functions form an additive basis of Sym and are ubiquitous in mathematics. Given partitions λ and μ, the product $s_\lambda \cdot s_\mu$ expands in the Schur basis with the same structure coefficients as the cup product expansion earlier.

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

While there are several combinatorial rules for computing LR coefficients, our motivation comes from geometric ways of computing them. Given partitions λ and μ, let $X_\lambda^{\mu^\vee}$ denote the Richardson variety obtained as the transversal intersection of the Schubert varieties X_λ and X_μ in $\mathrm{Gr}(k,n)$. Its cohomology class $[X_\lambda^{\mu^\vee}]$ equals $\sigma_\lambda \cup \sigma_\mu$ and hence LR coefficients can be calculated by studying its geometry. Starting from the work of Vakil [Vak06], this path has also been taken by Coskun [Cos09] and Liu [Liu10]. The core idea is to perform a series of flag degenerations that breaks the Richardson variety $X_\lambda^{\mu^\vee}$ into a union of Schubert varieties. All resulting algorithms have the feature that they can be diagramed by a rooted binary tree, where the root represents $X_\lambda^{\mu^\vee}$ and each node represents a component of the degeneration. It is in this setting that we have our results.

Consider the algebra of noncommutative symmetric functions, denoted by NSym, introduced in the seminal work [GKL$^+$95]. Since then, this algebra has come to play a major role in algebraic combinatorics, in no small measure due to its strong links with the algebras of symmetric functions and quasisymmetric functions. A distinguished basis for NSym, introduced by Bessenrodt, Luoto and van Willigenburg in [BLvW11] and whose origin can be traced to the theory of Macdonald polynomials, is the basis of noncommutative Schur functions $\mathbf{s}_\alpha$ where α is a strong composition. These functions are a noncommutative lift of Schur functions and exhibit many of their features such as Pieri rules, Kostka numbers, and an LR rule; we discuss the last of these next. The noncommutative LR coefficients $C^\gamma_{\alpha\beta}$ are defined as structure coefficients of the product

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_\gamma C^\gamma_{\alpha\beta}\, \mathbf{s}_\gamma$$

and they also turn out to be nonnegative integers. For more details on noncommutative Schur functions and their properties, the reader is referred to [LMvW13] and the numerous references therein.

In this extended abstract, we strengthen the connection between noncommutative Schur functions and classical Schur functions by identifying how noncommutative LR coefficients arise in Schubert calculus. More precisely, we

1. provide a rewrite of Coskun's Grassmannian algorithm (Table 1).

2. give a noncommutative generalization of Coskun's Grassmannian algorithm, and demonstrate how the original algorithm can be recovered as a special case (Table 2 and Theorem 4.7).

3. give geometric meaning to the noncommutative LR coefficients in the setting of the cohomology of the Grassmannian (Section 5 and Theorem 5.2).

Given the space constraints, we will omit proofs, but indicate proof techniques involved in Section 6 for the curious reader.

2 Background

By convention, given a positive integer n, we refer to the set of first n positive integers by $[n]$. Moreover, given positive integers p and q, we refer to the set $\{i \mid p \leq i \leq q\}$ as $[p, q]$, and call it an *interval*.

2.1 Compositions and partitions

A *weak composition* $\alpha = (\alpha_1, \ldots, \alpha_l)$ is a finite ordered list of nonnegative integers. The α_i for $1 \leq i \leq l$ are called the *parts* of α, while l is called the *length* of α and is denoted by $\ell(\alpha)$. The *strong composition* underlying α, denoted by α^+, is obtained by removing parts of α that equal 0. From this point on, we will take composition to mean a weak composition only. The *size* of α, denoted by $|\alpha|$, is defined to be the sum of its parts. The corresponding *reverse composition diagram* of α is an array of left-justified cells where the i-th row has α_i cells. Here we use the English convention in which the rows are ordered from top to bottom.

A strong composition $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a *partition* if $\lambda_1 \geq \cdots \geq \lambda_l$ holds. For any composition α, there is an underlying partition $\widetilde{\alpha}$ obtained by arranging the parts of α in weakly decreasing order and omitting zeros. For a partition λ, its *conjugate partition*, denoted by λ^t, is obtained by switching the rows and columns of the reverse composition diagram of λ. For any partition λ, we impose the convention that $\lambda_i = 0$ for all $i > \ell(\lambda)$. This given we can define dual partitions.

Suppose that the reverse composition diagram corresponding to a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ fits in a $k \times (n-k)$ rectangle, that is, $\lambda_1 \leq n-k$ and $l \leq k$. Then the *dual partition* $\lambda^\vee$ is defined to be the partition $\widetilde{\gamma}$ where $\gamma = (n-k-\lambda_k, \ldots, n-k-\lambda_1)$. Note that the reverse composition diagram corresponding to $\lambda^\vee$ is also contained in a $k \times (n-k)$ rectangle. Henceforth, we denote the fact that λ is contained in a $k \times (n-k)$ rectangle succinctly by stating $\lambda \subseteq (n-k)^k$. We further remark that the notion of dual partition only makes sense if we have a bounding rectangle to begin with. Finally, we note that the reverse composition diagram corresponding to a partition is the same as the more commonly used notion of Young diagram, using the English convention.

Example 2.1 *The reverse composition diagram of* $\alpha = (2, 0, 4, 3, 6)$ *is shown below.*

The strong composition corresponding to α *is* $\alpha^+ = (2, 4, 3, 6)$. *The underlying partition is* $\widetilde{\alpha} = (6, 4, 3, 2)$ *and its conjugate is* $\widetilde{\alpha}^t = (4, 4, 3, 2, 1, 1)$. *Note that* $\widetilde{\alpha}$ *fits in a* 5×6 *rectangle. Hence* $\widetilde{\alpha}^\vee = \widetilde{\gamma}$ *where* $\gamma = (6-0, 6-2, 6-3, 6-4, 6-6)$. *Thus* $\widetilde{\alpha}^\vee = (6, 4, 3, 2)$ *as well.*

2.2 Operators on compositions

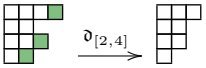
Next we consider two operators on compositions, whose definition is motivated from Pieri rules for quasisymmetric Schur functions and noncommutative Schur functions respectively.

The *box removing operators* on compositions $\mathfrak{d}_i$ for $i \geq 1$ have the following description: $\mathfrak{d}_i(\alpha)$ is the composition obtained by subtracting 1 from the rightmost part equaling i in α. If there is no such part, then $\mathfrak{d}_i(\alpha) = 0$.

Example 2.2 *Let* $\alpha = (2, 1, 2)$. *Then* $\mathfrak{d}_1(\alpha) = (2, 0, 2)$ *and* $\mathfrak{d}_2(\alpha) = (2, 1, 1)$.

Given a finite set $I = \{i_1 < \cdots < i_k\}$ of positive integers, define $\mathfrak{d}_I = \mathfrak{d}_{i_1}\mathfrak{d}_{i_2}\cdots\mathfrak{d}_{i_k}$. If I is the empty set, then we will think of $\mathfrak{d}_I$ as the identity map, and thus it does nothing to the composition it acts on.

Example 2.3 *If* $\alpha = (4,2,3,2)$, *then* $\mathfrak{d}_{[2,4]}(\alpha) = (3,2,2,1)$.



Consider now the *box adding operators* on compositions $\mathfrak{t}_i$ for $i \geq 1$ defined as follows: Given $\alpha = (\alpha_1, \ldots, \alpha_l)$, define $\mathfrak{t}_1(\alpha)$ to be $(1, \alpha_1, \ldots, \alpha_l)$ and $\mathfrak{t}_i(\alpha)$ to be $(\alpha_1, \ldots, \alpha_j + 1, \ldots, \alpha_l)$ if α_j is the leftmost part equaling $i-1$ in α for $i \geq 2$. If there is no such part, then $\mathfrak{t}_i(\alpha) = 0$.

Example 2.4 *Consider the composition* $\alpha = (3,2,3,1,2)$. *Then* $\mathfrak{t}_1(\alpha) = (1,3,2,3,1,2)$, $\mathfrak{t}_2(\alpha) = (3,2,3,2,2)$, $\mathfrak{t}_3(\alpha) = (3,3,3,1,2)$, $\mathfrak{t}_4(\alpha) = (4,2,3,1,2)$ *and* $\mathfrak{t}_i(\alpha) = 0$ *for all* $i \geq 5$.

2.3 Noncommutative Schur functions

The algebra of *noncommutative symmetric functions* NSym, introduced in [GKL$^+$95], is the free associative graded algebra generated over $\mathbb{Q}$ by the noncommuting indeterminates $\mathbf{h}_1, \mathbf{h}_2, \ldots$ where $\deg(\mathbf{h}_i) = i$ for $i \geq 1$. Given a strong composition $\alpha = (\alpha_1, \ldots, \alpha_l)$, we can define the *noncommutative homogeneous complete symmetric function* indexed by α, denoted by $\mathbf{h}_\alpha$, as follows.

$$\mathbf{h}_\alpha = \mathbf{h}_{\alpha_1} \cdots \mathbf{h}_{\alpha_l}$$

The set of $\mathbf{h}_\alpha$ where α runs over all strong compositions forms a multiplicative basis for NSym. As is the case in the classical theory of symmetric functions, NSym has a host of other interesting bases. Our interest lies in the basis of noncommutative Schur functions, denoted by $\mathbf{s}$, defined originally as being Hopf dual to the basis of quasisymmetric Schur functions. Here we adopt the following indirect inductive definition. For $n \geq 1$, we define $\mathbf{s}_{(n)}$ to be $\mathbf{h}_n$. Given a strong composition α that has at least two parts, we define $\mathbf{s}_\alpha$ using the left Pieri rule from [BLvW11].

Theorem 2.5 [BLvW11, Corollary 3.8] *Given positive integer* m *and a strong composition* β, *we have the following expansion.*

$$\mathbf{h}_m \cdot \mathbf{s}_\beta = \sum \mathbf{s}_\gamma$$

where the sum runs over all strong compositions γ *that can be written as* $\mathfrak{t}_{i_1} \cdots \mathfrak{t}_{i_m}(\beta) = \gamma$ *with* $i_1 > \cdots > i_m$.

Example 2.6 *The rule above implies that* $\mathbf{h}_1 \cdot \mathbf{s}_{(2)} = \mathbf{s}_{(1,2)} + \mathbf{s}_{(3)}$. *Using* $\mathbf{s}_{(2)} = \mathbf{h}_2$ *and* $\mathbf{s}_{(3)} = \mathbf{h}_3 = \mathbf{h}_{(3)}$, *we obtain the following expansion for* $\mathbf{s}_{(1,2)}$ *in the* $\mathbf{h}$*-basis.*

$$\mathbf{s}_{(1,2)} = \mathbf{h}_{(1,2)} - \mathbf{h}_{(3)}$$

The *projection map* χ from NSym to Sym mapping $\mathbf{h}_n$ to the complete homogeneous symmetric function h_n maps the noncommutative Schur function $\mathbf{s}_\alpha$ to the Schur function $s_{\widetilde{\alpha}}$ [BLvW11, Equation 2.12]. This can be seen by realizing that Theorem 2.5 reduces to the classical Pieri rule in the commutative setting. At this point, it is worth noting that the basis of Schur functions can also be defined indirectly as

the unique basis of Sym that satisfies the classical Pieri rule. Hence noncommutative Schur functions are indeed noncommutative lifts of Schur functions.

This brings us to our first result, which is a noncommutative analogue of the Jacobi-Trudi determinant formula for Schur functions. Since there is no uniform notion of determinants in noncommutative rings, we will adopt the following convention: Given an $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ where the a_{ij} are noncommuting indeterminates, we define $\det(A)$ to be

$$\sum_{\sigma \in S_n} sgn(\sigma) a_{\sigma(1),1} \cdots a_{\sigma(n),n}$$

where $sgn(\sigma)$ is the sign of the permutation σ. Using Theorem 2.5 in conjunction with an appropriate sign-reversing involution, we can prove the following.

Theorem 2.7 *Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be a partition. Consider the matrix $JT_\lambda = (\mathbf{h}_{\lambda_i - i + j})_{1 \leq i,j \leq l}$. Then $\mathbf{s}_\lambda = \det(JT_\lambda)$.*

In using the above theorem, we define $\mathbf{h}_0$ to be 1 and $\mathbf{h}_m = 0$ if $m < 0$.

Example 2.8 *Let $\lambda = (4,3,2)$. Then $JT_{(4,3,2)} = \begin{pmatrix} \mathbf{h}_4 & \mathbf{h}_5 & \mathbf{h}_6 \\ \mathbf{h}_2 & \mathbf{h}_3 & \mathbf{h}_4 \\ \mathbf{h}_0 & \mathbf{h}_1 & \mathbf{h}_2 \end{pmatrix}$ and Theorem 2.7 then implies the following expansion.*

$$\mathbf{s}_{(4,3,2)} = \mathbf{h}_{(4,3,2)} - \mathbf{h}_{(4,1,4)} - \mathbf{h}_{(2,5,2)} + \mathbf{h}_{(2,1,6)} + \mathbf{h}_{(5,4)} - \mathbf{h}_{(3,6)}$$

Theorem 2.5 is a special case of the noncommutative LR rule stated in [BLvW11, Theorem 3.5]. The *noncommutative LR coefficients* $C^\gamma_{\alpha\beta}$ are structure coefficients occurring in the following expansion.

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \sum_\gamma C^\gamma_{\alpha\beta}\, \mathbf{s}_\gamma$$

They turn out to be nonnegative integers, and are indeed given a combinatorial interpretation in [BLvW11]. For the purposes of this extended abstract, we do not state the aforementioned interpretation here, but mention that the $C^\gamma_{\alpha\beta}$ count the number of skew standard reverse composition tableaux of a fixed shape $\gamma /\!\!/ \beta$ that rectify to a unique standard reverse composition tableau of shape α.

This interpretation, though, is not easy to implement in practice. In what follows, not only do we give a straightforward algorithm to compute noncommutative LR coefficients, we further give the coefficients an interpretation in the setting of the cohomology of the Grassmannian in Section 5. The starting point for our proof is the noncommutative analogue of the Jacobi-Trudi formula stated in Theorem 2.7.

3 Geometric LR rules and the Coskun algorithm

In [Cos09], Coskun gives a geometric algorithm to compute classical LR coefficients using flag degenerations on the Richardson variety, and the combinatorial objects that keep track of these degenerations are LR Mondrian tableaux. We begin with the following alternate formulation of the original definition of LR Mondrian tableau [Cos09, Definition 3.22] for easier comprehension (with some work, this can be shown to be equivalent to Coskun's original definition).

Definition 3.1 *An* LR Mondrian tableau $M = (D_1, \ldots, D_s = AB_s, AB_{s+1}, \ldots, AB_k)$ *is an ordered collection of* k *distinct intervals in* $[n]$ *such that the following properties are satisfied.*

1. *If* $AB_i = [l_i, r_i]$ *and* $AB_{i+1} = [l_{i+1}, r_{i+1}]$, *then* $l_i < l_{i+1} \leq r_i + 1$ *and* $r_i < r_{i+1}$.
2. $|D_s \cup AB_{s+1}| - |AB_{s+1}| \geq s$.
3. *Either* $D_1 \subset \cdots \subset D_s$ *or there is a unique* D_t *such that the following hold.*
   - *(i)* $D_1 \subset \cdots \subset D_{t-1} \subset D_{t+1} \subset \cdots \subset D_s$ *and* $D_{t-1} \not\subseteq D_t \subset D_{t+1}$.
   - *(ii) For any* $u < t$, *if* $D_u = [l_u, r_u]$ *and* $D_t = [l_t, r_t]$, *then* $l_u < l_t$ *and* $r_u < r_t$.
   - *(iii)* $|D_{t-1} \cup D_t| - |D_t| \geq t - 1$, *and* $l_t \leq r_{t-1} + 1$.
   - *(iv)* $D_{t-1} \cup D_t$ *is a proper subset of* D_{t+1}.
   - *(v) If* $u < t < v \leq s$, *then* $l_u > l_v$.

We say an LR Mondrian tableau is nested *if* $D_1 \subset \cdots \subset D_s$ *and* $s = k$. *If* M *is not nested, then M has a unique* active interval *defined to be* D_{t-1} *if it exists and* D_s *otherwise.*

We call the two types of intervals appearing in the above definition the D intervals and the AB intervals. If the distinction between these two types of intervals is not required, then we will refer to the k intervals in an LR Mondrian tableau as $M_1, \ldots, M_k$.

Example 3.2 *Let* $k = 4$ *and* $n = 8$ *and consider the LR Mondrian tableau* $M = ([2,4], [1,5], [4,7], [6,8])$. *The* D *intervals are* $[2,4]$ *and* $[1,5]$, *while the* AB *intervals are* $[1,5]$, $[4,7]$ *and* $[6,8]$. *The active interval is* $[1,5]$. *Pictorially, we denote LR Mondrian tableaux by stacking the intervals of* M *in an array as shown below. Here we highlight the active interval.*

Given partitions $\lambda, \mu \subseteq (n-k)^k$, define the LR Mondrian tableau $M(\lambda, \mu)$ by defining the intervals $M_i(\lambda, \mu) := [n - k - \mu_i^\vee + i, n - k - \lambda_i + i]$. We state Coskun's recursive algorithm [Cos09, Algorithm 3.24] for Grassmannians in Table 1, where all the notation is as in Definition 3.1.

Coskun proves that both moves A and B map an LR Mondrian tableau M to another valid LR Mondrian tableau. He also proves that the conditions to apply at least one of moves always holds unless M is nested and hence the algorithm successfully terminates. Note that if M satisfies $r_i \geq l_{i+1} - 1$, for all $i \leq k$, then $A(M)$ and $B(M)$ also satisfy this condition for all $i \leq k$. Hence if $M(\lambda, \mu)$ satisfies this condition, then move A is always possible as the Coskun algorithm runs. It is easy to check that if $n - k \geq \lambda_1 + \mu_1$, then $r_i \geq l_{i+1} - 1$, for all $i \leq k$ in $M(\lambda, \mu)$. If M is nested, then we can associate a partition $\nu(M) = \widetilde{\gamma}$ where $\gamma = (n-k+1-|M_1| \geq n-k+2-|M_2| \geq \cdots \geq n-|M_k|)$. The output of Coskun's algorithm relates to the following theorem from [Cos09] which we have reformulated for the purpose of brevity.

Theorem 3.3 [Cos09, Theorem 3.25]. *Given partitions* λ, μ *such that* $\lambda, \mu \subseteq (n-k)^k$, *we have the following expansion.*

$$s_\lambda \cdot s_\mu = \sum_{M \in \mathrm{OutM}(\lambda,\mu)} s_{\nu(M)}$$

Input: Let $M := M(\lambda, \mu)$ and repeat the following moves until the algorithm terminates.

Move 0: If M is nested, then we are finished. Otherwise, let $M_i = D_s$ if the D intervals are nested, and if not, let $M_i = D_{t-1}$. We call M_i the *active* interval. Do the following moves if possible:

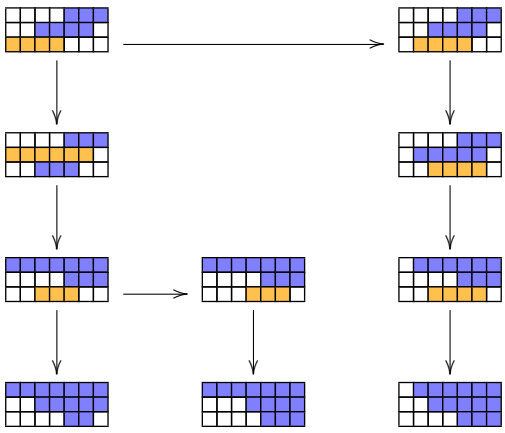
Move A: If $r_i \geq l_{i+1} - 1$, then replace the intervals M_{i+1}, M_i and M_u where $u < i$ and $l_u = l_i$ with
$M_{i+1} = [l_{i+1}, r_{i+1}] \mapsto A(M)_{i+1} := [l_i, r_{i+1}]$
$M_i = [l_i, r_i] \mapsto A(M)_i := [l_{i+1}, r_i + 1]$
$M_u = [l_u, r_u] \mapsto A(M)_u := [l_u + 1, r_u + 1]$
and fix all other intervals. Set $M := A(M)$.

Move B: If $l_i < l_{i+1} - i$ and $r_i < r_{i+1} - 1$, then replace the intervals M_i and M_u where $u < i$ and $l_u = l_i$ with
$M_i = [l_i, r_i] \mapsto B(M)_i := [l_i + 1, r_i + 1]$
$M_u = [l_u, r_u] \mapsto B(M)_u := [l_u + 1, r_u + 1]$
and fix all other intervals. Set $M := B(M)$.

Output: A finite collection of nested LR Mondrian tableaux.
Let $\mathrm{OutM}(\lambda, \mu)$ denote this multiset of LR Mondrian tableaux.

Tab. 1: Coskun algorithm [Cos09, Algorithm 3.24]

Example 3.4 *Let $k = 3, n = 7$ and $\lambda = (1), \mu = (2, 1)$. Coskun's algorithm starting from $M(\lambda, \mu) = ([1, 4], [3, 6], [5, 7])$ executes as shown below. Vertical and horizontal edges correspond to moves A and B respectively.*



From the above we conclude that

$$s_{(1)} \cdot s_{(2,1)} = s_{(3,1)} + s_{(2,2)} + s_{(2,1,1)}.$$

4 An algorithm for multiplying noncommutative Schur functions

In this section, we describe our algorithm for computing the product $\mathbf{s}_\alpha \cdot \mathbf{s}_\lambda$ where α is a strong composition and λ is a partition. This algorithm is similar to geometric LR rules found in [Cos09, Liu10, Vak06] in the sense that it can be represented by a rooted binary tree where the root corresponds to the input, the leaves correspond to the output and each edge corresponds to one iteration of the algorithm.

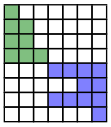
We call any rectangular grid B with shaded and empty cells a *board*. We use the standard matrix notation $\mathsf{B}(i,j)$ to reference the cell in the ith row and jth column of B where $\mathsf{B}(1,1)$ is the cell in the upper left corner. We say a shaded cell $\mathsf{B}(i,j)$ is *unstable* if the cell $\mathsf{B}(i,j-1)$ is empty. If the cell $\mathsf{B}(i,j-1)$ is shaded or $j=1$, then we say the shaded cell $\mathsf{B}(i,j)$ is *stable*. Note that only shaded cells can be stable or unstable. If $\mathsf{B}(i,j)$ is an unstable cell, then we *stabilize* the cell $\mathsf{B}(i,j)$ by shading in all empty cells to the left of $\mathsf{B}(i,j)$.

Example 4.1 *On the left is a board B with unstable cells $\mathsf{B}(1,5)$, $\mathsf{B}(2,5)$ and $\mathsf{B}(3,6)$, and on the right is the board obtained by stabilizing $\mathsf{B}(1,5)$.*

We say an unstable cell $\mathsf{B}(i,j)$ is *active* if for all other unstable cells $\mathsf{B}(k,l)$, we have $j \leq l$. For example, the board B above has two active cells at $\mathsf{B}(1,5)$ and $\mathsf{B}(2,5)$.

Definition 4.2 *Given strong compositions α and β satisfying $\ell(\alpha) \leq l$ and $\ell(\beta) \leq l$, we define the board $\mathsf{Init}(\alpha,\beta)$ as follows: Let the largest parts of α and β be i and j respectively. Place the reverse composition diagrams of α and β on a $2l \times (i+j)$ grid where we shade the diagram of α justified in the upper left corner and shade the diagram of β justified in the lower right corner.*

Example 4.3 *Let $\alpha=(1,2,2,3)$, $\beta=(4,2,4,1)$, and $l=4$. Then $\mathsf{Init}(\alpha,\beta)$ is the board below.*



It is easy to see that the number of unstable cells in $\mathsf{Init}(\alpha,\beta)$ is $\ell(\beta)$. Observe further that all cells corresponding to α are stable. We state our recursive noncommutative algorithm in Table 2.

To any stable board B, we can associate the corresponding weak composition $\gamma(\mathsf{B}) = (\gamma_1, \ldots, \gamma_{2l})$ where γ_i is the number of shaded cells in row i for $1 \leq i \leq 2l$. We now state one of our main results showing how the above algorithm computes noncommutative LR coefficients.

Theorem 4.4 *Given a strong composition α and a partition λ, we have the following expansion.*

$$\mathbf{s}_\alpha \cdot \mathbf{s}_\lambda = \sum_{\mathsf{B} \in \mathrm{Out}(\alpha,\lambda)} \mathbf{s}_{\gamma(\mathsf{B})^+}$$

Next we consider an example of the noncommutative algorithm. So as to get an idea of the results that follow, the reader is encouraged to compare it with Example 3.4 and notice the similarity.

Input: Let α be a strong composition and λ be a partition, and let $\mathsf{B} := \mathsf{Init}(\alpha, \lambda)$. Repeat the following moves until the algorithm terminates.

Move 0: If B has no unstable cells, then we are finished. Otherwise do the following moves if possible:

Move A: Move every active cell $\mathsf{B}(i, j+1)$ to $\mathsf{B}(i, j)$. Update B.

Move B: Let $\mathsf{B}(i, j+1)$ denote the active cell with the smallest row index and let q denote the number of empty cells to the left of $\mathsf{B}(i, j+1)$. If $\mathfrak{d}_{[j-q+1,j]}(\alpha) \neq 0$, then replace the reverse composition diagram of α with the reverse composition diagram of $\mathfrak{d}_{[j-q+1,j]}(\alpha)$ in the upper left corner and stabilize $\mathsf{B}(i, j+1)$. Update B.

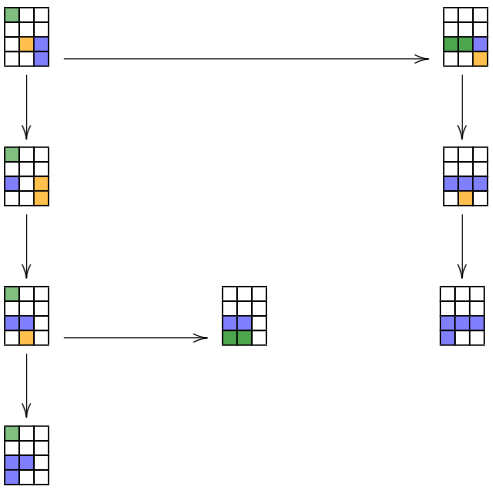
Output: A finite collection of stable boards, that is, boards with no unstable cells.
Let $\mathrm{Out}(\alpha, \lambda)$ denote this multiset of stable boards.

Tab. 2: Noncommutative algorithm

Example 4.5 *Let $\alpha = (1)$ and $\beta = (2, 1)$. Since both α and β fit inside a 2×2 rectangle, and the sum of the greatest parts of α and β is 3, our initial board has dimensions 4×3. Theorem 4.4 implies the following expansion, as the subsequent computation shows.*

$$\mathbf{s}_{(1)} \cdot \mathbf{s}_{(2,1)} = \mathbf{s}_{(1,2,1)} + \mathbf{s}_{(2,2)} + \mathbf{s}_{(3,1)}$$

Downward arrows correspond to a move A and rightward arrows correspond to a move B. Active cells are highlighted orange. Cells used to stabilize active cells are highlighted dark green.



Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, let $\lambda' = \xi^+$ where $\xi = (\lambda_1 - 1, \ldots, \lambda_l - 1)$. Then the binary tree generated by executing the noncommutative algorithm starting from $\mathsf{Init}(\alpha, \lambda')$ is isomorphic to a subtree of the one generated by $\mathsf{Init}(\alpha, \lambda)$. This recursiveness allows us to conclude the following.

Corollary 4.6 *(***Noncommutative scaling***) Given a strong composition α and a partition λ, let λ' be defined as before. If $\gamma = (\gamma_1, \ldots, \gamma_p)$ is a strong composition such that $C^{\gamma}_{\alpha\lambda} \neq 0$, then we have $C^{\gamma'}_{\alpha\lambda'} \neq 0$ where $\gamma' = \delta^+$ and $\delta = (\gamma_1 - 1, \ldots, \gamma_p - 1)$.*

Our next theorem shows that the Grassmannian algorithm of Coskun is a special case of the noncommutative algorithm. As a corollary of Theorem 4.4, we show both algorithms can be used to compute noncommutative LR coefficients in the case where α, β are partitions. One immediate consequence is a new proof that Coskun's algorithm computes classical LR coefficients.

Theorem 4.7 *Given partitions λ and μ, there is an explicit map Φ from LR Mondrian tableaux to boards which maps the rooted tree obtained from executing Coskun's algorithm starting from $M(\lambda^t, \mu^t)$ homeomorphically onto the rooted tree obtained from executing the noncommutative algorithm starting from initial board $\mathsf{Init}(\lambda, \mu)$. Moreover, this maps restricts to a bijection from $\mathrm{OutM}(\lambda^t, \mu^t)$ to $\mathrm{Out}(\lambda, \mu)$.*

The description of the map Φ, while not being convoluted, is omitted here for the sake of space. But we find it worthwhile to mention that our map links Coskun's and Liu's algorithms as well.

In view of the last statement of the above theorem, we can associate naturally a weak composition $\gamma(M)$ with a nested LR Mondrian tableau M. Then, as a corollary of the above theorem and Theorem 4.4, we obtain the following result that shows that Coskun's algorithm in fact computes certain noncommutative LR coefficients.

Corollary 4.8 *Given partitions λ and μ, we have the following expansion.*

$$\mathbf{s}_\lambda \cdot \mathbf{s}_\mu = \sum_{M \in \mathrm{OutM}(\lambda^t, \mu^t)} \mathbf{s}_{\gamma(M)}$$

In [Cos09, Theorem 3.25] and [Liu10, Theorem 3.1], Coskun and Liu show their algorithms compute classical LR coefficients. We give a new proof of their results using Corollary 4.8 and the following result from [BLvW11].

Theorem 4.9 ([BLvW11, Corollary 3.7]) *Given partitions λ, μ, ν and strong compositions α, β such that $\widetilde{\alpha} = \lambda$ and $\widetilde{\beta} = \mu$, we have the following equality.*

$$c^{\nu}_{\lambda\mu} = \sum_{\widetilde{\gamma} = \nu} C^{\gamma}_{\alpha\beta}$$

Corollary 4.10 *Given partitions λ, μ, and ν, we have the following equality.*

$$c^{\nu}_{\lambda\mu} = |\{M \in \mathrm{OutM}(\lambda, \mu) \mid \widetilde{\gamma(M)}^t = \nu\}|$$

5 Geometric interpretation of the noncommutative algorithm

Since Coskun's algorithm is derived using flag degenerations techniques, Corollary 4.8 gives geometric significance to noncommutative algorithm in the case where the input comprises of two partitions. We recall some preliminaries on Schubert calculus of the Grassmannian. Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ and let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k-dimensional vector subspaces in $\mathbb{C}^n$. For any partition $\lambda \subseteq (n-k)^k$ we define the *Schubert variety*

$$X_\lambda = \{V \in \mathrm{Gr}(k, n) \mid \dim(V \cap E_{n-k+i-\lambda_i}) \geq i, \ \forall\, 1 \geq i \geq k\}$$

where $E_j = \mathrm{span}\{e_1, \ldots, e_j\}$. Using dual partitions, we can also define the *opposite Schubert variety*

$$X^\lambda = \{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap \overline{E}_{n-k+i-\lambda_i^\vee}) \geq i,\ \forall\, 1 \geq i \geq k\}$$

where $\overline{E}_j = \mathrm{span}\{e_{n-j+1}, \ldots, e_{n-1}, e_n\}$. It is well known that

$$\dim(X^\lambda) = \mathrm{codim}(X_\lambda) = |\lambda|$$

and that the Schubert classes $\sigma_\lambda = [X_\lambda] = [X^{\lambda^\vee}]$ form an additive basis of the cohomology ring $H^*(\mathrm{Gr}(k,n))$. The *Richardson variety* is defined as the intersection $X_\lambda^\mu = X_\lambda \cap X^\mu$. This intersection is transverse and hence

$$[X_\lambda^{\mu^\vee}] = \sigma_\lambda \cup \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu\, \sigma_\nu$$

where the sum is taken over all partitions $\nu \subseteq (n-k)^k$. If we assume that $k \geq |\lambda|+|\mu|$ and $n-k \geq \lambda_1+\mu_1$, then the restriction that $\nu \subseteq (n-k)^k$ is trivial.

In Coskun's proof of the validity of his algorithm, he associates a certain variety to each Mondrian tableau as described next.

Definition 5.1 ([Cos09, Definition 3.27]) *Let $M = (M_1, \ldots, M_k)$ be an LR Mondrian tableau. For any $1 \leq i \leq j \leq k$, we define the vector space*

$$E_{M_i} = \mathrm{span}\{e_p \mid p \in M_i\}$$

and integers $r_{i,j} = \#\{M_l \subseteq (M_i \cap M_j)\}$. For any LR Mondrian tableau M we associate the subvariety $X_M \subseteq \mathrm{Gr}(k,n)$ defined as follows.

$$X_M = \{V \in \mathrm{Gr}(k,n) \mid \dim(V \cap E_{M_i} \cap E_{M_j}) \geq r_{i,j},\ \ \forall\, i \leq j\}$$

We remark that X_M is an irreducible subvariety of $\mathrm{Gr}(k,n)$ [Cos09, Lemma 3.28 and 3.29]. In fact, the variety associated with the initial LR Mondrian tableau $X_{M(\lambda,\mu)}$ is the Richardson variety $X_\lambda^{\mu^\vee}$. On the other hand, if M is a nested LR Mondrian tableau, then the variety X_M is isomorphic to the Schubert variety $X_{\nu(M)}$, and the cohomology class $[X_M]$ equals $\sigma_{\nu(M)}$ in $H^*(\mathrm{Gr}(k,n))$. As stated in Corollary 4.10, we have that $\nu(M) = \widetilde{\gamma(M)}^t$.

The LR Mondrian tableaux in the output of Coskun's algorithm record resulting varieties X_M that appear after performing a sequence of flag degenerations on the Richardson variety $X_\lambda^{\mu^\vee}$. Moves A and B in the algorithm describe these flag degenerations combinatorially. To put our noncommutative algorithm in this setting, we will need the notion of position of the variety X_M associated with a nested LR Mondrian tableau M. We say that the variety X_M is in *position* α if M is nested and $\gamma(M) = \alpha$. Note that the position of M is a weak composition. Given the above setup, we have the following result.

Theorem 5.2 *Let λ, μ, ν be partitions and γ be a strong composition such that $\widetilde{\gamma}^t = \nu$. Then the following equalities hold.*

$$C_{\lambda\mu}^\gamma = |\{M \in \mathrm{OutM}(\lambda^t, \mu^t) \mid X_M \textit{ is in position } \delta \textit{ where } \delta^+ = \gamma\ \}|$$
$$c_{\lambda\mu}^\nu = |\{M \in \mathrm{OutM}(\lambda^t, \mu^t) \mid \textit{the Schubert class } \sigma_\nu = [X_M]\}|$$

When calculating classical LR coefficients in Coskun's algorithm, the final step $X_M \mapsto [X_M]$ is analogous to applying the projection map χ to the noncommutative Schur function $\mathbf{s}_\gamma$ so as to obtain $s_{\widetilde{\gamma}} = \chi(\mathbf{s}_\gamma)$.

6 Remarks on the proof and algorithm for $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$

At this point, we would also like to emphasize the 'noncommutative nature' of our proof of the validity of the noncommutative algorithm. As mentioned earlier, we employ the left Pieri rule [BLvW11, Corollary 3.8] to prove our noncommutative Jacobi-Trudi determinantal analogue for $\mathbf{s}_\lambda$ where λ is a partition. Then using a Gessel-Viennot-type nonintersecting lattice paths argument involving an intricate sign-reversing involution, we express right multiplication by $\mathbf{s}_\lambda$ in terms of the action of certain monomials involving jeu de taquin (jdt) operators. The jdt operators have already been shown to arise naturally in the context of right Pieri rules for noncommutative Schur functions [Tew15], and our proof requires a careful study of the relations satisfied by these operators. The final step involves showing that the application of these jdt operators in a prescribed sequence is the same as picking a path from root to leaf in the rooted tree corresponding to the noncommutative algorithm. We close by revealing that we can compute efficiently any product of the form $\mathbf{s}_\alpha \cdot \mathbf{s}_\beta$ where α and β are strong compositions once we have computed $\mathbf{s}_\alpha \cdot \mathbf{s}_\lambda$ where $\lambda = \widetilde{\beta}$ using the noncommutative algorithm.

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