Cyclic inclusion-exclusion and the kernel of P-partitions

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Abstract. Following the lead of Stanley and Gessel, we consider a linear map which associates to an acyclic directed graph (or a poset) a quasi-symmetric function. The latter is naturally defined as multivariate generating series of non-decreasing functions on the graph (or of P-partitions of the poset).

We describe the kernel of this linear map, using a simple combinatorial operation that we call *cyclic inclusion-exclusion*. Our result also holds for the natural non-commutative analog and for the commutative and non-commutative restrictions to bipartite graphs.

Résumé. Dans la lignée de Stanley et Gessel, nous considérons une application linéaire qui associe à un graphe dirigé acyclique (ou à un poset) une fonction quasi-symétrique. Celle-ci est définie comme la série génératrice multi-variée des fonction croissantes sur le graphe (ou des P-partitions du poset).

Nous décrivons le noyau de cette application linéaire, à l'aide d'une opération combinatoire simple, que nous appelons *inclusion-exclusion cyclique*. Notre résultat est aussi valable pour l'analogue non-commutatif naturel et les restrictions commutative et non-commutative aux graphes bipartis.

Keywords. partially ordered sets, P-partitions, quasi-symmetric functions

This is an extended abstract of the paper [4], that shall be published elsewhere.

1 Introduction

Given an acyclic directed graph $G = (V, E_G)$, it is natural to consider the following multivariate generating function

$$\Gamma_G(x_1, x_2, \cdots) = \sum_{\substack{f:V\to\mathbb{N} \\ f \text{ non-decreasing}}} \prod_{v\in V} x_{f(v)} \tag{1}$$

where $\mathbb{N}$ is the set of positive integers and *non-decreasing* means that $(i, j) \in E_G$ implies $f(i) \le f(j)$. An example is given in Section 2.4. A similar definition can be considered for a poset $P = (V, <_P)$, replacing $(i, j) \in E_G$ by $i <_P j$.

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This is a classical object in the algebraic combinatorics literature: using the terminology of the seminal book of Stanley [10], the non-decreasing functions on posets correspond to P-partitions when P has a *natural labelling* (up to reversing the order of P). The generating function Γ_P has then been considered by Gessel [6], see also Stanley's textbook [11, Section 7.19]. While not symmetric in the variables $x_1, x_2, \dots$, the function Γ_P exhibits some weaker symmetry property and belongs to the now well-studied algebra of *quasi-symmetric functions* – in fact, the terminology *quasi-symmetric function* was introduced in [6], precisely to study Γ_P.

Although posets are more common objects in the literature, the results of this paper are better formulated in terms of acyclic directed graphs. Obviously the map $\Gamma : G \to \Gamma_G$ defined by (1) can be extended by linearity to the vector space $\mathcal{G}$ of formal linear combinations of acyclic directed graphs, that we call here the *graph algebra*. We refer the reader to [5] for a study of this map (and an extension) from a Hopf algebra point of view. Here, we only focus on the linear structure.

The main result of the present paper is a combinatorial description of the kernel of the map Γ from the graph algebra to quasi-symmetric functions (Theorem 2). This description relies on a simple combinatorial operation, that we call *cyclic inclusion-exclusion* (the definition and an example are given in Section 3.1). Before giving some background on this operation, let us mention that this description of the kernel of Γ is quite robust. Indeed, we shall prove that cyclic inclusion-exclusion also describes the kernel of some variants of Γ, as follows:

- Working with *labeled* (acyclic directed) graphs, it is natural to associate to them a multivariate generating series in *non-commuting variables*. This object lives in the algebra of *word quasi-symmetric functions* [9], sometimes also called *quasi-symmetric functions in non-commuting variables*; see [1]. We give a description of the kernel of this map (denoted Γ^{nc}) in Theorem 1.
- In the long version of this paper [4], we also consider restrictions of the linear maps Γ and Γ^{nc} to bipartite graphs and describe the kernel of these restrictions via cyclic inclusion-exclusion.

In all these cases, a byproduct of our proof is the surjectivity of the morphism Γ (respectively Γ^{nc} and their restriction to bipartite graphs). The surjectivity in the commutative non-restricted case was observed by Stanley [12, Note p7], answering a question of Billera and Reiner.

Our proofs use a combination of basic linear algebra, graph combinatorics and (word) quasi-symmetric function manipulations. In the non-commutative/labeled case, we first exhibit a family of graphs so that their images form a $\mathbb{Z}$-basis of word quasi-symmetric functions. Then, we show that these graphs span the quotient of the graph algebra by cyclic inclusion-exclusion relations. With an easy linear algebra argument, this concludes the proof.

The commutative/unlabeled case can be obtained as a corollary of the non-commutative/labeled case. In contrast, restrictions to bipartite graphs must be considered separately from the non-restricted setting. The general structure of the proof is the same in the bipartite setting, although the arguments themselves are quite different. Because of the limited space in this extended abstract, we do not present the bipartite case here.

Along the way, this gives natural bases of the word quasi-symmetric function ring. These new bases are non-commutative lifts of known bases of the quasi-symmetric function ring: in particular, we find natural analogs of Gessel fundamental basis [6] and of two bases introduced respectively by R. Stanley in [12] and by K. Luoto in [8].

Let us now say a word about the cyclic inclusion-exclusion operation and how it has proved useful so far. It has been introduced by the author in the article [3] in the proof of a conjecture of Kerov on irreducible character values of the symmetric group. The question addressed here (whether cyclic inclusion-exclusion relations do or do not span the kernel of Γ) is natural in this context, since it allows to simplify some arguments in the proof. We refer the reader to the long version [4, Section 6] for details.

Remarkably, this operation of cyclic inclusion-exclusion has also been fruitful in a quite different context in [2], where the purpose was to study some rational functions introduced by Greene [7]. These functions are indexed by posets and defined as sums over linear extensions of the indexing poset: as such, they automatically verify cyclic inclusion-exclusion relations. This gives an efficient way to compute these rational functions and a powerful tool to study them; see [2].

2 Preliminaries

2.1 Labelled and unlabeled graphs

Throughout the paper we work with simple directed graphs, both labeled and unlabeled.

Definition 2.1 *A* labeled (simple directed) graph *G is a pair (V, E) where V is a finite set and E a subset of $V \times V$. A* directed cycle *is a list $(v_1, \dots, v_k)$ of vertices of G such that (v_1, v_2), (v_2, v_3), $\cdots$, (v_{k-1}, v_k) and (v_k, v_1) are edges of G. A graph without directed cycles is called* acyclic.

An unlabeled graph is an equivalence class of labeled graphs, with respect to the *relabeling* operation.

2.2 Quasi-symmetric functions

The ring of quasi-symmetric functions was introduced by I. Gessel [6] and may be seen as an extension of symmetric functions. A modern introduction can be found in [11, Section 7.19].

Let n be a non-negative integer. A *composition* (or *integer composition*) of n is a list $I = (i_1, i_2, \dots, i_r)$ of positive integers, whose sum is equal to n. The notation $I \vDash n$ means that I is a composition of n and $\ell(I)$ denotes the number of parts of I. In numerical examples, it is customary to omit parentheses and commas. For example, 212 is a composition of 5.

Consider the algebra $\mathbb{C}[[X]]$ of formal power series in a totally ordered countable set of commutative variables $X = \{x_1, x_2, \dots\}$. Monomials $X^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} \cdots$ correspond to sequences $\mathbf{v} = v_1, v_2, \dots$ with finitely many non-zero entries. For such a sequence, we denote by $\mathbf{v}_{\leftarrow}$ the finite list obtained by omitting the zeroes in $\mathbf{v}$.

Definition 2.2 *A formal power series $f \in \mathbb{C}[[X]]$ is said to be* quasi-symmetric *if and only if*

- *there exists m such that each monomial with a nonzero coefficient in f has degree at most m;*
- *and, for any $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v}_{\leftarrow} = \mathbf{w}_{\leftarrow}$, the coefficients of $X^{\mathbf{v}}$ and $X^{\mathbf{w}}$ in P are equal.*

One can easily prove that the set of quasi-symmetric functions is a subalgebra of $\mathbb{C}[[X]]$, called the quasi-symmetric function ring and denoted $QSym$.

It should be clear that any symmetric function is quasi-symmetric. The algebra $QSym$ of quasi-symmetric functions has a basis of monomial quasi-symmetric functions (M_I) indexed by compositions $I = (i_1, \dots, i_r)$, where

$$M_I = \sum_{k_1 < \cdots < k_r} x_{k_1}^{i_1} \cdots x_{k_r}^{i_r}. \tag{2}$$

Example 2.3 $M_{212} = \sum_{k<l<m} x_k^2 x_l x_m^2$.

2.3 Word quasi-symmetric functions

The natural non-commutative analog of $QSym$ is the algebra of *word quasi symmetric functions*, denoted by $\mathbf{WQSym}$. We recall here its construction, following the presentation of Bergeron and Zabrocki [1, Section 5.2]. An equivalent, but slightly different presentation, using packed words instead of set-compositions, can be found in a paper of Novelli and Thibon [9, Section 2.1].

Consider a totally ordered countable alphabet of non-commuting variables $A = \{a_1, a_2, \dots\}$. We denote by $\mathbb{C}\langle\langle A\rangle\rangle$ the algebra of formal power series in the set of variables A. Monomials in A are canonically indexed by finite words w on the alphabet $\mathbb{N}$ as follows $\boldsymbol{a}_w = a_{w_1}\, a_{w_2} \dots a_{w_{|w|}}$. The *evaluation* $\mathrm{eval}(w)$ of a word w is the integer sequence $v = (v_1, v_2, \dots)$, where v_i is the number of letters i in w. Then the commutative image of $\boldsymbol{a}_w$ is $\boldsymbol{X}^{\mathrm{eval}(w)}$.

In the non-commutative framework, set-compositions play the role of compositions. A *set-composition* of n is a list $\mathbf{I} = (I_1, \dots, I_p)$ of pairwise disjoint non-empty subsets of $\{1, \dots, n\}$, whose union is $\{1, \dots, n\}$ (in the literature, set-compositions are sometimes called *ordered set partitions*). In numerical example, we sort integers inside a part and use a vertical line to separate the parts. For example, the set-composition $(\{1, 5\}, \{3, 4, 6\}, \{2\})$ is denoted $15|346|2$.

To a word w on the (ordered) alphabet $\mathbb{N}$ of length ℓ, we associate the set-composition $\mathbf{I} = \Delta(w)$ such that $j \in I_{|\{w_r : w_r \leq w_j\}|}$ (for every j in $[\ell]$). In other words, I_1 contains the positions of the smallest letter in w, I_2 the positions of the second smallest, and so on. For example, $\Delta(275525) = 15|346|2$.

Definition 2.4 *A formal power series f in $\mathbb{C}\langle\langle A\rangle\rangle$ is a* word quasi symmetric function *if and only if*
- *there exists m such that each monomial with a nonzero coefficient in f has degree at most m;*
- *and the coefficients of a_v and a_w in f are equal as soon as $\Delta(v)$ and $\Delta(w)$ coincide.*

One can easily prove that the set $\mathbf{WQSym}$ of word quasi symmetric functions is an algebra. A linear basis of $\mathbf{WQSym}$ is given as follows:

$$\mathbf{M_I} = \sum_{w \text{ s.t.} \Delta(w) = \mathbf{I}} \boldsymbol{a}_w.$$

Note that sending the variables $a_1, a_2, \dots$ to their commutative analogs $x_1, x_2, \dots$ defines an algebra morphism from $\mathbf{WQSym}$ to $QSym$.

Example 2.5 *Consider the set-composition $\boldsymbol{I} = 25|4|13$. Then the associated basis element of* $\mathbf{WQSym}$ *is*

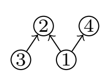
$$\mathbf{M_I} = \sum_{k<l<m} a_m\, a_k\, a_m\, a_l\, a_k.$$

Clearly, its commutative image is M_{212} (given in Example 2.3).

2.4 Gessel's morphism

Definition 2.6 *Let G be a graph on vertex set $[n]$. A function $f : [n] \to \mathbb{N}$ is called G* non-decreasing *if, for any edge (i, j) in E, one has $f(i) \leq f(j)$. For a labeled graph G, we define $\Gamma^{nc}(G)$ as*

$$\Gamma^{nc}(G) := \sum_{\substack{f : [n] \to \mathbb{N} \\ f \ G \text{ non-decreasing}}} a_{f(1)} \dots a_{f(n)}.$$

Example 2.7 *Consider the graph $G =$* 

, then $\Gamma^{nc}(G) = \sum_{\substack{k_1, k_2, k_3, k_4 \\ k_3 \leq k_2,\ k_1 \leq k_2,\ k_1 \leq k_4}} a_{k_1} a_{k_2} a_{k_3} a_{k_4}.$

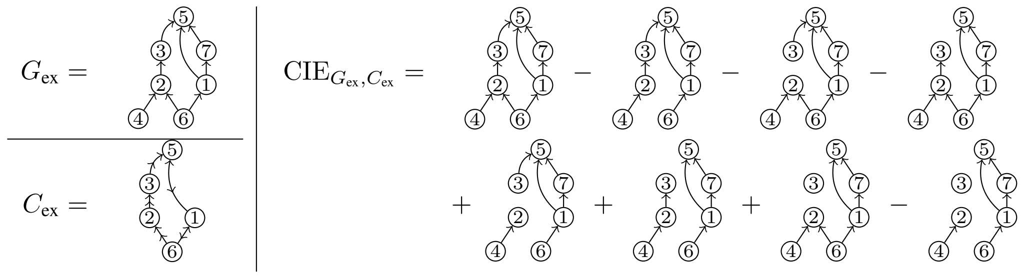


Fig. 1: Graph G_{ex}, cycle C_{ex} and the graph algebra element $\text{CIE}_{G_{\text{ex}},C_{\text{ex}}}$ from Example 3.1.

It is clear that $\Gamma^{\text{nc}}(G)$ is a word quasi-symmetric function. Therefore, Γ^{nc} extends to a linear map from $\mathscr{G}$ to $\mathbf{WQSym}$. The map Γ defined by Eq. (1) in the introduction and introduced by Gessel in [6] is a quotient of Γ^{nc}, replacing the variables $a_1, a_2, \dots$ by their commutative analogs $x_1, x_2, \dots$

Remark 2.8 *The map Γ^{nc}, and hence its quotient Γ, are Hopf algebra morphisms (with a suitable Hopf algebra structure on $\mathcal{G}$). Indeed, this is the restriction to posets of the map $\Gamma_{(1,1,1)}$ introduced by Foissy and Malvenuto in [5].*

3 Cyclic inclusion-exclusion

3.1 Definition and example

Let G be a directed graph. Assume that, as an undirected graph, it contains a cycle C. Formally, such a cycle C is a list $C = (x_1, \dots, x_k)$ such that, for $1 \le i \le k$, either (x_i, x_{i+1}) is an edge of G, or (x_{i+1}, x_i) is an edge of G (where, by convention, $x_{k+1} := x_1$). In the first case, we say that (x_i, x_{i+1}) is in C^+. In the second case, we say that (x_i, x_{i+1}) is in C^-.

Another description of the sets C^+ and C^- is the following. Edges of C have two orientations: their orientation in the cycle C and their orientation as edges of G. We denote C^+ (respectively C^-) the set of edges of C, for which these two orientations coincide (respectively do not coincide).

Finally, for a subset D of edges of G, we denote by $G \setminus D$ the (directed acyclic) graph obtained from G by erasing the edges in D (and keeping the same set of vertices). Then, in the graph algebra $\mathscr{G}$, we set

$$\text{CIE}_{G,C} = \sum_{D \subseteq C^+} (-1)^{|D|} G \setminus D.$$

Example 3.1 *Consider the graph G_{ex} from Figure 1. The non-oriented version of G_{ex} contains several cycles, among them $C_{ex} = (6, 2, 3, 5, 1)$. This cycle is represented as a subgraph of G_{ex} in Figure 1 with the two edge orientations described above. Then the set C^+_{ex} is equal to $\{(6, 2), (2, 3), (3, 5)\}$ and $\text{CIE}_{G_{ex},C_{ex}}$ is given in Figure 1.*

3.2 Cyclic inclusion-exclusion relations

Proposition 3.2 *For any graph G and cycle C of G, one has:*

$$\Gamma^{nc}(\text{CIE}_{G,C}) = 0.$$

Proof: Let n be the size of G. From the definitions, one has:

$$\Gamma^{\text{nc}}(\text{CIE}_{G,C}) = \sum_{D \subseteq C^+} (-1)^{|D|} \left[\sum_{\substack{f:[n]\to\mathbb{N} \\ f \ (G\setminus D) \text{ non-decreasing}}} a_{f(1)} \cdots a_{f(n)} \right]$$

$$= \sum_{f:[n]\to\mathbb{N}} \big(a_{f(1)} \cdots a_{f(n)}\big) \left(\sum_{D \subseteq C^+} (-1)^{|D|} \big[f \ (G \setminus D) \text{ non-decreasing} \big] \right),$$

where [condition] is 1 if the condition is fulfilled and 0 else.

The idea of the proof is to show that for any function $f : [n] \to \mathbb{N}$, one has

$$\sum_{D \subseteq C^+} (-1)^{|D|} \big[f \ (G \setminus D) \text{ non-decreasing} \big] = 0. \tag{3}$$

If f is not a $G \backslash C^+$ non-decreasing function, then each summand of (3) is zero and the conclusion holds trivially in this case. Consider now a $G \backslash C^+$ non-decreasing function $f : [n] \to \mathbb{N}$. For such a function f, we define $D_f = \big\{ (x,y) \in C^+ \text{ s.t. } f(x) > f(y) \big\}$. It is straightforward that D_f fulfills the following property:

$$\forall D \subseteq C^+, \quad f \text{ is } G \setminus D \text{ non-decreasing} \iff D_f \subseteq D. \tag{4}$$

Hence the left-hand side of (3) can be rewritten as $\sum_{D_f \subseteq D \subseteq C^+} (-1)^{|D|}$, which is equal to zero if and only if $D_f \neq C^+$. Therefore, we need to show that, for any $G \backslash C^+$ non-decreasing function, D_f is strictly included in C^+.

We proceed by contradiction. Suppose that we can find a $G \backslash C^+$ non-decreasing function f for which $D_f = C^+$. This means that, for each (x,y) in C^+, one has $f(x) > f(y)$. Besides, since f is a $G \backslash C^+$ non-decreasing function, one has $f(x) \leq f(y)$ for any edge (x,y) of G which is not in C^+. Recall now that C is a cycle in the undirected version of G. Formally, C is a list $(x_1, \dots, x_k)$ such that, for $1 \leq i \leq k$, (by convention, $x_{k+1} = x_1$)

- either (x_i, x_{i+1}) is an edge of G and $(x_i, x_{i+1}) \in C^+$;
- or (x_{i+1}, x_i) is an edge of G and $(x_i, x_{i+1}) \in C^-$.

Using the remarks above, we can conclude in both cases that $f(x_i) \geq f(x_{i+1})$. Bringing everything together,

$$f(x_1) \geq f(x_2) \geq \cdots \geq f(x_\ell - 1) \geq f(x_\ell) \geq f(x_1).$$

As C^+ cannot be empty (otherwise, $(x_k, \dots, x_1)$ would be a directed cycle), at least one of these inequalities should be strict. We have reached a contradiction and D_f must be strictly included in C^+, which ends the proof of the proposition. □

Proposition 3.2 gives some relations between the word quasi-symmetric functions $\Gamma^{\text{nc}}(G)$. We call these relations *cyclic inclusion-exclusion relations* (CIE relations for short). Formally, the elements $(\text{CIE}_{G,C})$ span a subspace of $\mathscr{G}$, that we shall denote $\mathscr{C}$, which is included in the kernel of Γ^{nc}.

We shall prove in the next section that any relation among the $\Gamma^{\text{nc}}(G)$ can be deduced from CIE relations. In other words, the space $\mathscr{C}$ is exactly the kernel of Γ^{nc}. We will also prove a similar result for its commutative quotient Γ.

Remark 3.3 *With the definition above, a cycle can go several times through the same vertex, or even contain several copies of the same edge. We could also forbid this in the definition of cycles, the results of the paper would be identical.*

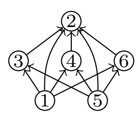
Remark 3.4 *A weaker form of Proposition 3.2 (in the commutative setting) has been established in [2, Theorem 4.1]. The structure of the proof is identical.*

4 The kernel of Gessel's morphism

4.1 The graphs $G_{\mathbf{I}}$

Definition 4.1 *Let $\mathbf{I} = (I_1, \ldots, I_r)$ be a set-composition of $[n]$. We consider the directed graph $G_{\mathbf{I}}$ with vertex set $[n]$ and edge set $\bigsqcup_{j<k} I_j \times I_k$. In other words, there is an edge between x and y if the index of the set of $\mathbf{I}$ containing x is smaller than the one of the set-containing y.*

Example 4.2 *Take $\mathbf{I}_{ex} = 15|346|2$. Then $G_{\mathbf{I}_{ex}}$ and the associated word quasi symmetric function are*

$G_{\mathbf{I}_{ex}} =$ 

$$; \quad \Gamma^{nc}(G_{\mathbf{I}_{ex}}) = \sum_{\substack{k_1,\ldots,k_6 \\ \max(k_1,k_5)\leq\min(k_3,k_4,k_6) \\ \max(k_3,k_4,k_6)\leq k_2}} a_{k_1} \cdots a_{k_6}. \qquad (5)$$

4.2 A $\mathbb{Z}$-basis of $\mathbf{WQSym}$

The purpose of this Section is to prove that $\Gamma^{\text{nc}}(G_{\mathbf{I}})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. The proof requires to consider two additional bases of $\mathbf{WQSym}$ and to observe that three change of basis matrices are unitriangular (with respect to different orders of the basis elements). It will be more convenient for us to work with descent-starred permutations, instead of set-compositions.

Definition 4.3 *We define a* descent-starred permutation *as a couple (σ, D) such that D is a subset of the descent set $\{i; \sigma(i) > \sigma(i+1)\}$ of the permutation σ. The descents in D are termed* starred.

In numerical example, we represent a descent-starred permutation (σ, D) by the word notation of σ in which the elements of index in D are followed by a star. For example the descent-starred permutation $(3142, \{3\})$ will be denoted $314_\star 2$.

Lemma 4.4 *Descent-starred permutations of n are in bijection with set-compositions of $[n]$.*

Proof: From the numerical notation of a set-composition $\mathbf{I}$, we sort each part in decreasing order and remove vertical bars to get the word notation of σ. Then mark with a star the descents inside the same part of $\mathbf{I}$. For example, the descent-starred permutation associated to $15|346|2$ is $5_\star 16_\star 4_\star 32$. This is clearly a bijection. □

Let us define two families of word quasi-symmetric functions indexed by descent-starred permutations: $\mathbf{M}_{(\sigma,D)}$ and $\mathbf{F}_{(\sigma,D)}$. Both are defined as a sum $\sum a_{k_1} \cdots a_{k_n}$ over lists $\mathbf{k} = (k_1, \ldots, k_n)$ of positive integers with conditions given in the table below (for integers x in $[n-1]$).

	$\mathbf{M}_{(\sigma,D)}$	$\mathbf{F}_{(\sigma,D)}$
$x \in D$	$k_{\sigma(x)} = k_{\sigma(x+1)}$	$k_{\sigma(x)} < k_{\sigma(x+1)}$
$x \notin D$	$k_{\sigma(x)} < k_{\sigma(x+1)}$	$k_{\sigma(x)} \leq k_{\sigma(x+1)}$

Let us focus on $\mathbf{M}_{(\sigma,D)}$. We require that $k_{\sigma(x)} = k_{\sigma(x+1)}$ for $x \in D$, which implies that the function $x \mapsto k_x$ should be constant on the parts of the associated set-composition $\mathbf{I}$. Moreover, the conditions $k_{\sigma(x)} = k_{\sigma(x+1)}$ for $x \in D$ and $k_{\sigma(x)} < k_{\sigma(x+1)}$ for $x \notin D$ are equivalent to $\Delta(\mathbf{k}) = \mathbf{I}$, so that we have $\mathbf{M}_{(\sigma,D)} = M_{\mathbf{I}}$.

Remark 4.5 *The commutative projection of* $\mathbf{F}_{(\sigma,D)}$ *is* F_J*, where* F *is the* fundamental basis *of* $QSym$ *and* J *the (integer) composition associated with the set* D *(we use here the terminology of [11, Section 7.19]).*

Lemma 4.6 *The family* $(\mathbf{F}_{(\sigma,D)})$*, indexed by descent-starred permutations, is a* $\mathbb{Z}$*-basis of* $\mathbf{WQSym}$*.*

Proof: Omitted: the argument relies on triangular changes of basis, introducing an intermediate family $(\mathbf{L}_{(\sigma,D)})$. See the long version [4] for details. □

We now explain how $\Gamma^{\text{nc}}(G_{\mathbf{I}})$ expands on the $\mathbf{F}$ basis. If $\mathbf{I} = (I_1, \dots, I_r)$ is a set-composition, we consider the following set $\mathrm{DSP}(\mathbf{I})$ of descent-starred permutations:

- As a word $\sigma = w^1 \cdots w^r$, where w^m is a permutation of I_m ;
- The descent in position x is starred if σ_x and σ_{x+1} are in the same part of $\mathbf{I}$. In other words, for each m, we mark the descents in w^m, but not the potential descent created by concatenating w^m and w^{m+1}.

For example, take $\mathbf{I}_{\text{ex}} = 15|346|2$, then $\mathrm{DSP}(\mathbf{I}_{\text{ex}})$ contains the following 12 descent-starred permutations:

$$153462,\ 5_\star13462,\ 154_\star362,\ 5_\star14_\star362,\ 156_\star4_\star32,\ 5_\star16_\star4_\star32,\ 1536_\star42,\ 5_\star136_\star42,\ 1546_\star32,\ 5_\star146_\star32,\ 156_\star342,\ 5_\star16_\star342.$$

Proposition 4.7 *For any set-composition* $\mathbf{I}$*, one has:*

$$\Gamma^{nc}(G_{\mathbf{I}}) = \sum_{(\sigma,D)\in \mathrm{DSP}(\mathbf{I})} \mathbf{F}_{(\sigma,D)}.$$

Proof: Let (σ, D) be a descent-starred permutation in $\mathrm{DSP}(\mathbf{I})$ and $a_{k_1} \cdots a_{k_n}$ a monomial in $\mathbf{F}_{(\sigma,D)}$. Consider j_1 in I_m and j_2 in I_{m+1} for some m. By definition of $\mathrm{DSP}(\mathbf{I})$, clearly, j_1 appears before j_2 in σ, which implies that $k_{j_1} \leq k_{j_2}$. Therefore $j \mapsto k_j$ is a $G_{\mathbf{I}}$ non-increasing function and the monomial $a_{k_1} \cdots a_{k_n}$ also appears in $\Gamma^{\text{nc}}(G_{\mathbf{I}})$.

Conversely, we have to prove that any monomial in $\Gamma^{\text{nc}}(G_{\mathbf{I}})$ appears in exactly one of the functions $\mathbf{F}_{(\sigma,D)}$, for (σ, D) in $\mathrm{DSP}(\mathbf{I})$. Let f be a $G_{\mathbf{I}}$ non-decreasing function from $[n]$ to $\mathbb{N}$. We want to construct (σ, D) in $\mathrm{DSP}(\mathbf{I})$ such that, for all x between 1 and $n-1$, one has

$$f\big(\sigma(x)\big) \leq f\big(\sigma(x+1)\big), \text{ with strict inequalities for } x \text{ in } D. \tag{6}$$

By definition of $\mathrm{DSP}(\mathbf{I})$, σ should be a concatenation $w^1 \cdots w^r$, where each w^m is a permutation of I_m. Moreover, the potential descent between w^m and w^{m+1} is not starred. Since f is $G_{\mathbf{I}}$ non-decreasing, the inequality (6) holds when $\sigma(x)$ and $\sigma(x+1)$ lie respectively in I_m and I_{m+1} for some m (by definition of $\mathrm{DSP}(\mathbf{I})$, in this case, $x \notin D$). Therefore, we focus on the case where $\sigma(x)$ and $\sigma(x+1)$ are in the same part I_m of I.

Let us consider the restriction f_m of f to I_m. We claim that there exists a unique word w^m which is a permutation of I_m and such that $f_m(w^m_i) \leq f_m(w^m_{i+1})$, with strict inequality whenever i is a descent of

w^m. Indeed this word is obtained by ordering lexicographically the pairs $\big((f_m(y), y)\big)_{y \in I_m}$ and keeping only the second element of each pair.

We mark the descents in w^m and by concatenating all the words w^m (for $1 \leq m \leq r$), we get a descent-starred permutation (σ, D) in $\mathrm{DSP}(\mathbf{I})$. By construction, this descent-starred permutation is the unique one in $\mathrm{DSP}(\mathbf{I})$ such that $a_{f(1)} \cdots a_{f(n)}$ appears in $\mathbf{F}_{(\sigma,D)}$, which ends the proof. □

Example 4.8 *Take $\mathbf{I}_{ex}$ as in Example 4.2, $\Gamma^{nc}(G_{\mathbf{I}_{ex}})$ is given by Eq.* (5)*. The summation set can be split as follows:*

- *either $k_1 \leq k_5$ or $k_5 < k_1$;*
- *besides, the integers k_3, k_4 and k_6 fulfill exactly one of the 6 following inequalities:*

$$k_3 \leq k_4 \leq k_6, \quad k_4 < k_3 \leq k_6, \quad k_3 \leq k_6 < k_4,$$

$$k_4 \leq k_6 < k_3, \quad k_6 < k_3 \leq k_4, \quad k_6 < k_4 < k_3.$$

Combining both case distinctions yield 12 different cases, and $\Gamma^{nc}(G_{\mathbf{I}_{ex}})$ is a sum of 12 different terms which are the $\mathbf{F}$ functions indexed by the 12 descent-starred permutations in $\mathrm{DSP}(\mathbf{I}_{ex})$ (which are listed above).

Corollary 4.9 *The family $\big(\Gamma^{nc}(G_{\mathbf{I}})\big)$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$.*

Proof: If (σ, D) is the descent-starred permutation associated by Lemma 4.4 to a set-composition $\mathbf{I}$ of n of length r, then the size of D is $n - r$. Besides, for each element $(\sigma', D') \in \mathrm{DSP}(\mathbf{I})$, the size of D' is smaller than $n - r$, unless $(\sigma', D') = (\sigma, D)$. Hence Proposition 4.7 implies that the matrix of $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ in the basis $\mathbf{F}_{(\sigma,D)}$ is unitriangular with respect to the order $(\sigma', D') <_2 (\sigma, D) \Leftrightarrow |D'| < |D|$ and $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ is a $\mathbb{Z}$-basis of $\mathbf{WQSym}$. □

Remark 4.10 *Stanley fundamental theorem on P-partitions [10, Theorem 6.2] implies that, if G is a naturally labeled graph (*i.e.* such that $(i, j) \in E$ implies $i \leq j$ as positive integers), then $\Gamma^{nc}(G)$ has a non-negative expansion on the $\mathbf{F}_{(\sigma,D)}$ basis. Proposition 4.7 gives examples of non-necessarily naturally labeled graphs G, such that the $\mathbf{F}_{(\sigma,D)}$ expansion of $\Gamma^{nc}(G)$ has non-negative coefficients. But this is not the case for any graph G, as shown by the following example (we skip details in the computation):*

$$\Gamma^{nc}\left(\begin{smallmatrix} ① & \\ \uparrow & ② \\ ③ & \end{smallmatrix}\right) = \mathbf{F}_{231} + \mathbf{F}_{3_\star 2_\star 1} + \mathbf{F}_{312} - \mathbf{L}_{3_\star 2_\star 1} = \mathbf{F}_{231} + \mathbf{F}_{312} + \mathbf{F}_{3_\star 21} + \mathbf{F}_{32_\star 1} - \mathbf{F}_{321}.$$

Such negative signs do not occur in the commutative setting: indeed, any function $\Gamma(\overline{G})$ is a non-negative linear combination of fundamental quasi-symmetric functions, see [11, Corollary 7.19.5].

4.3 A generating family for the quotient

We will now show that $(G_{\mathbf{I}})$, where $\mathbf{I}$ runs over all set-compositions, spans the quotient $\mathscr{G}/\mathscr{C}$. Here is the key combinatorial lemma in this section.

Lemma 4.11 *Let G be an unlabeled acyclic directed graph. Then either G is equal to some $G_{\mathbf{I}}$ or, in the quotient $\mathscr{G}/\mathscr{C}$, one can write G as a linear combination of graphs with the same set of vertices and more edges.*

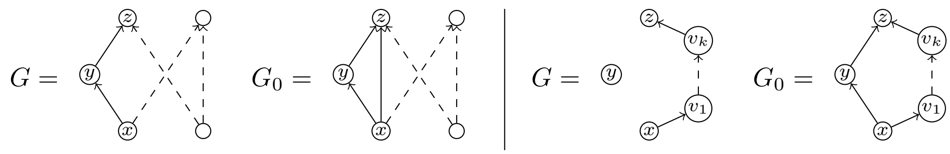


Fig. 2: On the left (resp. right): graphs G and G_0 in the first (resp. second) case of the proof of Lemma 4.11.

Proof: Let G be an acyclic directed graph with vertex set $[n]$ and edge set E_G. Throughout the proof, we denote $\sim$ the following symmetric relation: $x \sim y$ if, in G, there is no directed path from x to y, nor from y to x. When $x \sim y$, the directed graphs $G_{(x,y)}$ and $G_{(y,x)}$ obtained from G by adding respectively an edge from x to y or from y to x are still acyclic. We distinguish three cases.

Case 1: G is not the graph of a transitive relation.

In other words, there exist x, y and z such that

- there is an edge from x to y and from y to z in G;
- there is no edge from x to z.

We consider $G_0 = G_{(x,z)}$ the graph obtained from G by adding an edge between x and z. As a directed graph, G_0 is acyclic: otherwise, there would be a path from z to x in G and, together with (x, y) and (y, z), this path would be a directed cycle in G. But the non-oriented version of G_0 contains a cycle $C = (x, z, y)$. Using the notation of Section 3.1, one has $C^+ = \{(x, z)\}$ and the corresponding cyclic inclusion-exclusion element is $\mathrm{CIE}_{G_0,C} = G_0 - G$. Hence, in $\mathscr{G}/\mathscr{C}$, one has $G = G_0$ and the statement is true in this case.

This case is illustrated on the left-hand side of Figure 2 with examples of graphs G and G_0. Dashed edges are edges of G and G_0 that do not play a role in the proof.

Case 2: the relation $\sim$ is not an equivalence relation.

By assumption, there exist vertices x, y, z such that

- there is a path $(x, v_1, \cdots, v_k, z)$ from x to z in G;
- one has $x \sim y$ and $y \sim z$.

By definition of $\sim$, the graph $G_{(x,y)}$ is acyclic. Moreover, it does not contain a path from z to y. Indeed, as $y \sim z$ in G, such a path should use the edge (x, y) and thus be the concatenation of a path from z to x with the edge (x, y). But G does not contain a path from z to x (indeed, it contains a path from x to z and no directed cycles).

Therefore, the graph G_0 obtained from $G_{(x,y)}$ by adding an edge from y to z is an acyclic directed graph. Then the undirected version of G_0 contains a cycle $C = (x, y, z, v_k, \cdots, v_1)$. Using the notation of Section 3.1, for this cycle, one has $C^+ = \{(x, y), (y, z)\}$. Hence,

$$\mathrm{CIE}_{G_0,C} = G_0 - G_0 \setminus \{(x, y)\} - G_0 \setminus \{(y, z)\} + G_0 \setminus \{(x, y), (y, z)\}.$$

But $G_0 \setminus \{(x, y), (y, z)\}$ is G, so, in the quotient $\mathscr{G}/\mathscr{C}$, one has

$$G = -G_0 + G_0 \setminus \{(x, y)\} + G_0 \setminus \{(y, z)\}$$

and the statement is proved in this case.

This case is illustrated on the right-hand side of Figure 2 with examples of graphs G and G_0. Here, the dashed edge illustrates the fact that the length of the path P from x to z can be arbitrary. Potential extra edges and vertices of G and G_0 have not been represented for more readability.

Case 3: G is the graph of a transitive relation and the relation $\sim$ is an equivalence relation.

In this case, we will prove that G is necessarily equal to $G_{\mathbf{I}}$, for some set-composition $\mathbf{I}$. Let us start by a remark: in the graph of a transitive relation, the existence of a path from x to y implies the existence of an edge from x to y. Hence $x \nsim y$ means that there is either an edge from x to y or from y to x. Denote by $(V_j)_{j \in J}$ the partition of the vertex set of G into equivalence classes of $\sim$. Consider two such classes V_j and V_k. We will prove that either $V_j \times V_k$ or $V_k \times V_j$ is included in E_G.

Select arbitrarily a pair (v_0, w_0) in $V_j \times V_k$. As $v_0 \nsim w_0$, by eventually swapping v_0 and w_0 (and simultaneously j and k), we may assume that (v_0, w_0) is an edge of G. Then, for any w in V_k, the pair (v_0, w) is also an edge of G. Indeed, if this is not the case, since $v_0 \nsim w$, this would imply that (w, v_0) is an edge of V. But, then by transitivity, (w, w_0) should be an edge of G, which is impossible since $w \sim w_0$.

The same argument proves that, for any v in V_j, the pair (v, w) must be an edge of G, which proves the inclusion of $V_j \times V_k$ in E_G. As we may have swapped j and k after selecting v_0 and w_0, we have in fact proved that for any pair (j, k) in J^2, either $V_j \times V_k$ or $V_k \times V_j$ is included in E_G. Since G does not have any directed cycle, there exists a total order $<_J$ on J such that $V_j \times V_k$ is included in E_G if and only if $j <_J k$.

By definition of $\sim$, there is no edges with both extremities in the same V_j. Besides, there can not be an edge from V_k to V_j (with $j <_J k$), as this would create a directed cycle of length 2. Finally, the set of edges of G is exactly $\bigsqcup_{j <_J k} V_j \times V_k$, which means that $G = G_{\mathbf{I}}$ for $\mathbf{I} = (V_j)_{j \in J}$. □

Let G be an acyclic directed graph. Iterating Lemma 4.11, one can write G as an integer linear combination of $G_{\mathbf{I}}$ in the quotient space $\mathscr{G}/\mathscr{C}$ (since we are working with simple graphs, this iteration always terminates). In other words, $G_{\mathbf{I}}$ spans the vector space $\mathscr{G}/\mathscr{C}$.

4.4 First main result

We are now ready to prove the following statement.

Theorem 1 *The space $\mathscr{C}$, spanned by cyclic inclusion-exclusion elements, is the kernel of the surjective morphism Γ^{nc} from $\mathscr{G}$ to $\mathbf{WQSym}$.*

Proof: Denote by $\mathscr{K}$ the kernel of Γ^{nc}. By Proposition 3.2, it contains $\mathscr{C}$. On one hand (Section 4.3), we know that $\mathscr{G}/\mathscr{C}$ is spanned by the family $(G_{\mathbf{I}})$. On the other hand (Corollary 4.9), the family $\Gamma^{\mathrm{nc}}(G_{\mathbf{I}})$ is a basis of $\mathbf{WQSym}$, which implies in particular that the $(G_{\mathbf{I}})$ are linearly independent in $\mathscr{G}/\mathscr{K}$ and hence in $\mathscr{G}/\mathscr{C}$. Therefore $(G_{\mathbf{I}})$ is a basis of $\mathscr{G}/\mathscr{C}$ and Γ^{nc} is an isomorphism from $\mathscr{G}/\mathscr{C}$ to $\mathbf{WQSym}$ (it sends a basis on a basis), which concludes the proof. □

Remark 4.12 *Note that the proof of Lemma 4.11 only uses cyclic inclusion-exclusion with $|C^+| = 1$ or $|C^+| = 2$. Therefore, we have in fact proved a stronger result: the subspace of $\mathscr{G}$ spanned by cyclic inclusion-exclusion associated to cycles C with $|C^+| = 1$ and $|C^+| = 2$ is the kernel of Γ (and hence coincides with $\mathscr{C}$).*

4.5 Unlabeled commutative framework and second main result

Consider an unlabeled directed graph $\overline{G}$ and a cycle $\overline{C}$ of the undirected version of $\overline{G}$. As in Section 3.1, we can define a cyclic inclusion-exclusion element in the unlabeled graph algebra $\overline{\mathscr{G}}$. Alternatively, $\overline{\mathrm{CIE}}_{\overline{G},\overline{C}}$ is the image of $\mathrm{CIE}_{G,C}$ by the quotient morphism $\mathscr{G} \to \overline{\mathscr{G}}$.

Let us consider the subspace $\overline{\mathscr{C}}$ of $\overline{\mathscr{G}}$ spanned by cyclic inclusion-exclusion elements.

Theorem 2 *The ideal $\overline{\mathscr{C}}$, spanned by inclusion-exclusion elements, is the kernel of the surjective morphism Γ from $\overline{\mathscr{G}}$ to $QSym$.*

Proof: This follows from Theorem 1, since $QSym$ and $\overline{\mathscr{G}}$ are quotients of $\mathbf{WQSym}$ and $\mathscr{G}$, respectively, and since the maps Γ and Γ^{nc} are compatible with the quotient structure; see the long version [4] for details. □

Remark 4.13 *The function $\Gamma\left(\overline{G_{\mathbf{I}}}\right)$ in $QSym$ depends only on the integer composition I obtained from $\mathbf{I}$ by keeping only the sizes of the parts. Therefore, from Section 4.2, we know that the $\mathbb{Z}$-span of this family is $QSym$. As it is indexed by integer compositions, this family is a $\mathbb{Z}$-basis of $QSym$. This family has appeared in a paper of Stanley [12, Note p7] who noticed that the change of basis matrix with the fundamental basis is unitriangular (commutative version of Proposition 4.7).*

References

[1] N. Bergeron and M. Zabrocki. The Hopf algebras of symmetric functions and quasisymmetric functions in non-commutative variables are free and cofree. *J. of Algebra and its Applications*, 8(4):581–600, 2009.

[2] A. Boussicault and V. Féray. Application of graph combinatorics to rational identities of type A. *Elec. Jour. Combinatorics*, 16(1):R145, 2009.

[3] V. Féray. Combinatorial interpretation and positivity of Kerov's character polynomials. *J. Algebr. Comb*, 29(4):473–507, 2009.

[4] V. Féray. Cyclic inclusion-exclusion. arXiv:1410.1772, 2014.

[5] L. Foissy and C. Malvenuto. The Hopf algebra of finite topologies and T-partitions. *Journal of Algebra*, 438:130–169, 2015.

[6] I. Gessel. Multipartite P-partitions and inner products of Schur functions. *Contemp. Math*, 34:289–302, 1984.

[7] C. Greene. A rational function identity related to the Murnaghan-Nakayama formula for the characters of S_n. *J. Algebr. Comb.*, 1(3):235–255, 1992.

[8] K. Luoto. A matroid-friendly basis for the quasisymmetric functions. *Journal of Combinatorial Theory, Series A*, 115(5):777–798, 2008.

[9] J.-C. Novelli and J.-Y. Thibon. Polynomial realizations of some trialgebras. *FPSAC proceedings*, pages 243–255, 2006.

[10] R. Stanley. *Ordered structures and partitions*, volume 119 of *Memoirs of the Amer. Math. Soc.* 1972.

[11] R. Stanley. *Enumerative combinatorics, Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

[12] R. Stanley. The descent set and connectivity set of a permutation. *Journal of Integer Sequences*, 8(2):3, 2005.