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The flag upper bound theorem for 3- and 5-manifolds

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Abstract. We prove that among all flag 3-manifolds on n vertices, the join of two circles with $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ vertices respectively is the unique maximizer of the face numbers. This solves the first case of a conjecture due to Lutz and Nevo. Further, we establish a sharp upper bound on the number of edges of flag 5-manifolds and characterize the cases of equality. We also show that the inequality part of the flag upper bound conjecture continues to hold for all flag 3-dimensional Eulerian complexes and characterize the cases of equality in this class.

Résumé. Nous montrons que parmi tous les 3-variétés à n sommets, le join de deux cercles avec $\lceil \frac{n}{2} \rceil$ et $\lfloor \frac{n}{2} \rfloor$ sommets respectivement est la seule variété qui maximise le nombre de faces. Cela prouve le premier cas d'une conjecture due à Lutz et Nevo. Par ailleurs, nous établissons une borne supérieure optimale pour le nombre d'arêtes des 5-variétés de cliques et nous caractérisons les cas d'égalité. Nous montrons également que l'inégalité de la conjecture de la borne supérieure est satisfaite pour toutes les complexes de cliques Eulériens 3-dimensionnels et nous caractérisons les cas d'égalité dans cette classe.

Keywords. Upper bound theorem, Flag complex, Eulerian complex, Simplicial manifold

1 Introduction

One of the classical problems in geometric combinatorics deals with the following question: for a given class of simplicial complexes, find tight upper bounds on the number of i-dimensional faces as a function of the number of vertices and the dimension. Since Motzkin (1957) proposed the upper bound conjecture (UBC, for short) for polytopes in 1957, this problem has been solved for various families of complexes. In particular, McMullen (1970) and Stanley (1975) proved that neighborly polytopes simultaneously maximize all the face numbers in the class of polytopes and simplicial spheres. However, it turns out that, apart from cyclic polytopes, many other classes of neighborly spheres or even neighborly polytopes exist, see Shermer (1982) and Padrol (2013) for examples and constructions of neighborly polytopes.

A simplicial complex Δ is *flag* if all of its minimal non-faces have cardinality two, or equivalently, Δ is the clique complex of its graph. Flag complexes form a beautiful and important class of simplicial complexes. For example, barycentric subdivisions of simplicial complexes, order complexes of posets, and Coxeter complexes are flag complexes. Despite a lot of effort that went into studying the face numbers

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of flag spheres, in particular in relation with the Charney-Davis conjecture (Charney and Davis, 1995), and its generalization given by Gal's conjecture (Gal, 2005), a flag upper bound theorem for spheres is still unknown. The upper bounds of face numbers for general simplicial $(d-1)$-spheres are far from sharp for those of flag $(d-1)$-spheres, since the graph of any flag $(d-1)$-dimensional complex is K_{d+1}-free. Gal (2005) established the upper bound theorem for flag spheres of dimension less than five. However, starting from dimension five, there are only conjectural upper bounds. For $m \geq 1$, we let $J_m(n)$ be the $(2m-1)$-sphere on n vertices obtained as the join of m copies of the circle, each one a cycle with either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$ vertices. We also let $J^*_m(n)$ be the $2m$-sphere on n vertices defined as the suspension of $J_m(n-2)$.

Conjecture 1.1 *(Nevo and Petersen, 2010, Conjecture 6.3) If Δ is a flag homology sphere, then $\gamma(\Delta)$ satisfies the Frankl-Füredi-Kalai inequalities on $\lfloor \frac{\dim \Delta + 1}{2} \rfloor$-colored complexes. In particular, if Δ is of dimension $2m-1$, where $m \geq 2$, then $f_i(\Delta) \leq f_i(J_m(n))$ for $1 \leq i \leq 2m-1$; if Δ is of dimension $2m$, where $m \geq 1$, then $f_i(\Delta) \leq f_i(J^*_m(n))$ for $1 \leq i \leq 2m$.*

As for the case of equality, Lutz and Nevo posited that as opposed to the case of all simplicial spheres, for a fixed dimension $2m-1$ and the number of vertices n, there is *only* one maximizer of the face numbers.

Conjecture 1.2 *Lutz and Nevo (2014, Conjecture 6.3) Let $m \geq 2$ and Δ be a flag simplicial $(2m-1)$-sphere on n vertices. Then $f_i(\Delta) = f_i(J_m(n))$ for some $1 \leq i \leq m$ if and only if $\Delta = J_m(n)$.*

Recently, Adamaszek and Hladký (2015) proved that this conjecture holds asymptotically for flag homology manifolds. Several celebrated theorems from extremal graph theory served as tools for their work. As a result, the proof simultaneously gives upper bounds on f-numbers, h-numbers, g-numbers and γ-numbers, but it only applies to flag homology manifolds with an extremely large number of vertices.

Our first main result is that both Conjecture 1.1 and Conjecture 1.2 hold for *all* flag 3-manifolds.

Theorem 1.3 *Let Δ be a flag 3-manifold on n vertices. Then $f_i(\Delta) \leq f_i(J_2(n))$. If equality holds for some $1 \leq i \leq 3$, then $\Delta = J_2(n)$.*

The proof of Theorem 1.3 only relies on simple properties of flag complexes and Eulerian complexes. We also establish an analogous result on the number of edges of flag 5-manifolds.

Theorem 1.4 *Let Δ be a flag homology 5-manifold on n vertices. Then $f_1(\Delta) \leq f_1(J_3(n))$. Equality holds if and only if $\Delta = J_3(n)$.*

In 1964, Klee (1964) proved that Motzkin's UBC for polytopes holds for a much larger class of Eulerian complexes as long as they have sufficiently many vertices, and conjectured that the UBC holds for *all* Eulerian complexes. Our second main result deals with flag Eulerian complexes, and asserts that Conjecture 1.1 continues to hold for all 3-dimensional flag Eulerian complexes.

Theorem 1.5 *Let Δ be a 3-dimensional flag Eulerian complex on n vertices. Then $f_i(\Delta) \leq f_i(J_2(n))$ for $1 \leq i \leq 3$.*

This provides supporting evidence to a question of Adamaszek and Hladký (2015, Problem 17(i)) in the case of dimension 3, where they proposed that Conjecture 1.1 holds for all odd-dimensional flag weak pseudomanifolds with sufficiently many vertices. We also give constructions of the maximizers of face numbers in this class and show that they are the only maximizers. Our proof is based on an application of the inclusion-exclusion principle and double counting.

The Extended Abstract is organized as follows. In Section 2, we discuss basic facts on simplicial complexes and flag complexes. In Section 3, we provide the proof of our first main result asserting that given a number of vertices n, the maximum face numbers of a flag 3-manifold are achieved only when this manifold is the join of two circles of length as close as possible to $\frac{n}{2}$. In Section 4, we apply an analogous argument to the class of flag 5-manifolds. In Section 5, we show that the same upper bounds continue to hold for the class of 3-dimensional flag Eulerian complexes, and discuss the maximizers of the face numbers in this class. Finally, we close in Section 6 with some concluding remarks.

2 Preliminaries

A *simplicial complex* Δ on a vertex set $V = V(\Delta)$ is a collection of subsets $\sigma \subseteq V$, called faces, that is closed under inclusion. For $\sigma \in \Delta$, let $\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximal dimension of its faces. A *facet* in Δ is a maximal face under inclusion, and we say that Δ is *pure* if all of its facets have the same dimension.

If Δ is a simplicial complex and σ is a face of Δ, the *link* of σ in Δ is $\mathrm{lk}_\Delta\, \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *deletion* of a vertex set W from Δ is $\Delta \backslash W := \{\sigma \in \Delta : \sigma \cap W = \emptyset\}$. The *restriction* of Δ to a vertex set W is defined as $\Delta[W] := \{\sigma \in \Delta : \sigma \subseteq W\}$. If Δ and Γ are two simplicial complexes on disjoint vertex sets, then the *join* of Δ and Γ, denoted as $\Delta * \Gamma$, is the simplicial complex on vertex set $V(\Delta) \cup V(\Gamma)$ whose faces are $\{\sigma \cup \tau : \sigma \in \Delta, \tau \in \Gamma\}$.

A simplicial complex Δ is a *simplicial manifold* (resp. *simplicial sphere*) if the geometric realization of Δ is homeomorphic to a manifold (resp. sphere). We denote by $\tilde{H}_*(\Delta; \mathbf{k})$ the reduced homology of Δ computed with coefficients in a field $\mathbf{k}$, and by $\beta_i(\Delta; \mathbf{k}) := \dim_{\mathbf{k}} \tilde{H}_i(\Delta; \mathbf{k})$ the reduced Betti numbers of Δ with coefficients in $\mathbf{k}$. We say that Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology manifold* if $\tilde{H}_*(\mathrm{lk}_\Delta\, \sigma; \mathbf{k}) \cong \tilde{H}_*(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every nonempty face $\sigma \in \Delta$. A $\mathbf{k}$-*homology sphere* is a $\mathbf{k}$-homology manifold that has the $\mathbf{k}$-homology of a sphere. Every simplicial manifold (resp. simplicial sphere) is a homology manifold (resp. homology sphere). Moreover, in dimension two, the class of homology 2-spheres coincides with that of simplicial 2-spheres, and hence in dimension three, the class of homology 3-manifolds coincides with that of simplicial 3-manifolds.

For a $(d-1)$-dimensional complex Δ, we let $\chi(\Delta) := \sum_{i=0}^{d-1} (-1)^i \beta_i(\Delta; \mathbf{k})$ be the *reduced Euler characteristic* of Δ. A simplicial complex Δ is called an *Eulerian* complex if Δ is pure and $\chi(\mathrm{lk}_\Delta\, \sigma) = (-1)^{\dim \mathrm{lk}_\Delta \sigma}$ for every $\sigma \in \Delta$, including $\sigma = \emptyset$. In particular, it follows from the Poincaré duality theorem that all odd-dimensional simplicial manifolds are Eulerian.

A $(d-1)$-dimensional simplicial complex Δ is called a *weak* $(d-1)$-*pseudomanifold* if it is pure and every $(d-2)$-face (called *ridge*) of Δ is contained in exactly two facets. A weak $(d-1)$-pseudomanifold Δ is called a *normal* $(d-1)$-*pseudomanifold* if it is connected, and the link of each face of dimension $\leq d-3$ is also connected. Every Eulerian complex is a weak pseudomanifold, and every connected homology manifold is a normal pseudomanifold. In fact, every normal 2-pseudomanifold is also a homology 2-manifold. However, for $d > 3$, the class of normal $(d-1)$-pseudomanifolds is much larger than the class of homology $(d-1)$-manifolds. It is well-known that if Δ is a weak (resp. normal) $(d-1)$-pseudomanifold and σ is a face of Δ of dimension at most $d-2$, then the link of σ is also a weak (resp. normal) pseudomanifold. The following lemma gives another property of normal pseudomanifolds, see Bagchi and Datta (2008, Lemma 1.1).

Lemma 2.1 *Let Δ be a normal $(d-1)$-pseudomanifold, and let W be a subset of vertices of Δ such that* *the induced subcomplex $\Delta[W]$ is a normal $(d-2)$-pseudomanifold. Then the induced subcomplex of Δ on vertex set $V(\Delta)\backslash W$ has at most two connected components.*

For a $(d-1)$-dimensional complex Δ, we let $f_i = f_i(\Delta)$ be the number of i-dimensional faces of Δ for $-1 \le i \le d-1$. The vector $(f_{-1}, f_0, \cdots, f_{d-1})$ is called the *f-vector* of Δ. Since the graph of any simplicial 2-sphere is a maximal planar graph, it follows that the f-vector of a simplicial 2-sphere is uniquely determined by f_0. For a 3-dimensional Eulerian complex, the following lemma indicates that its f-vector is uniquely determined by f_0 and f_1. (We omit the proof.)

Lemma 2.2 *The f-vector of a 3-dimensional Eulerian complex satisfies*

$$(f_0, f_1, f_2, f_3) = (f_0, f_1, 2f_1 - 2f_0, f_1 - f_0).$$

A simplicial complex Δ is *flag* if all minimal non-faces of Δ, also called missing faces, have cardinality two; equivalently, Δ is the clique complex of its graph. The following lemma (Nevo and Petersen, 2010, Lemma 5.2) gives a basic property of flag complexes:

Lemma 2.3 *Let Δ be a flag complex on vertex set V. If $W \subseteq V$, then $\Delta[W]$ is also flag. Furthermore, if σ is a face in Δ, then $\operatorname{lk}_\Delta \sigma = \Delta[V(\operatorname{lk}_\Delta \sigma)]$. In particular, all links in a flag complex are also flag.*

Finally, we recall some terminology from graph theory. A graph G is a path graph if the set of its vertices can be ordered as $x_1, x_2, \cdots, x_n$ in such a way that $\{x_i, x_{i+1}\}$ is an edge for all $1 \le i \le n-1$ and there are no other edges. Similarly, a cycle graph is a graph obtained from a path graph by adding an edge between the end points of the path.

3 The Proof of flag UBC for flag 3-manifolds

Recall that in the introduction, we defined $J_m(n)$ to be the $(2m-1)$-sphere on n vertices obtained as the join of m circles, each one of length either $\lfloor \frac{n}{m} \rfloor$ or $\lceil \frac{n}{m} \rceil$. The goal of this section is to prove the flag UBC for flag 3-manifolds (see Conjectures 1.1 and 1.2). We start with the following lemma.

Lemma 3.1 *Let Δ be a flag normal 3-pseudomanifold on n vertices. Then $f_1(\Delta) \le f_1(J_2(n)) + c$, where $c = 3 - 3\min_{v\in\Delta} \chi(\operatorname{lk}_\Delta v)$.*

Proof: Let v be a vertex of maximum degree in $V(\Delta)$. We let $a = f_0(\operatorname{lk}_\Delta v)$, $W_1 = V(\operatorname{lk}_\Delta v)$ and $W_2 = V(\Delta)\backslash V(\operatorname{lk}_\Delta v)$. Since Δ is a normal 3-pseudomanifold, $\operatorname{lk}_\Delta v$ is a normal 2-pseudomanifold, i.e., a simplicial 2-manifold. Furthermore, since Δ is flag, by Lemma 2.3, $\operatorname{lk}_\Delta v$ is the restriction of Δ to W_1. Thus, by Lemma 2.1, the induced subcomplex $\Delta[W_2]$ has at most two connected components. Since v is not connected to any vertices in $W_2\backslash\{v\}$, it follows that $\{v\}$ and $\Delta[W_2\backslash\{v\}]$ are the two connected components in $\Delta[W_2]$.

We now count the edges of Δ. They consist of the edges of $\Delta[W_1] = \mathrm{lk}_\Delta\, v$, the edges of $\Delta[W_2]$ and the edges between these two sets. In addition, $\sum_{w\in W_2} f_0(\mathrm{lk}_\Delta\, w)$ counts the edges of $\Delta[W_2]$ twice. Thus,

$$\begin{aligned} f_1(\Delta) &= f_1(\Delta[W_1]) + \Big(\sum_{w\in W_2} f_0(\mathrm{lk}_\Delta\, w)\Big) - f_1(\Delta[W_2]) \\ &\overset{(*)}{\leq} f_1(\mathrm{lk}_\Delta\, v) + |W_2| \cdot \max_{w\in W_2} f_0(\mathrm{lk}_\Delta\, w) - (f_0(\Delta[W_2\backslash\{v\}]) - 1) \\ &\overset{(**)}{=} \big(3a - 6 + 3(1-\chi(\mathrm{lk}_\Delta\, v))\big) + (n-a)a - (n-a-2) \\ &= -a^2 + a(n+4) - (n+4) + 3 - 3\chi(\mathrm{lk}_\Delta\, v) \\ &\overset{(***)}{\leq} \left\lfloor \frac{n^2}{4} \right\rfloor + n + 3 - 3\chi(\mathrm{lk}_\Delta\, v) \\ &= f_1(J_2(\Delta)) + 3(1-\chi(\mathrm{lk}_\Delta\, v)). \end{aligned} \tag{1}$$

Here in (*) we used that $\Delta[W_2\backslash\{v\}]$ is connected and hence has at least $f_0(\Delta[W_2\backslash\{v\}]) - 1$ edges. Equality (**) follows from the fact that $\mathrm{lk}_\Delta\, v$ is a 2-manifold with a vertices, and (***) is obtained by optimizing the function $p(a) = -a^2 + a(n+4)$. Hence the result follows. □

Proof of Theorem 1.3: We use the same notation as in the proof of Lemma 3.1. That is, we let v be a vertex of maximum degree in $V(\Delta)$. We let $a = f_0(\mathrm{lk}_\Delta\, v)$, $W_1 = V(\mathrm{lk}_\Delta\, v)$ and $W_2 = V(\Delta)\backslash V(\mathrm{lk}_\Delta\, v)$. Since Δ is a flag 3-manifold, $\chi(\mathrm{lk}_\Delta\, w) = 1$ for every $w \in \Delta$. Hence by Lemma 3.1, $f_1(\Delta) \leq f_1(J_2(\Delta))$. Furthermore, it follows from steps (*) and (***) in equality (1) that $f_1(\Delta) = f_1(J_2(n))$ holds only if $f_0(\mathrm{lk}_\Delta\, w) = a = \lceil \frac{n+4}{2} \rceil$ or $\lfloor \frac{n+4}{2} \rfloor$ for all $w \in W_2$, and $\Delta[W_2\backslash\{w\}]$ is a tree.

We claim that if $f_1(\Delta) = f_1(J_2(n))$, then $\Delta = J_2(n)$. This indeed holds if $n = 8$ or 9, since the only flag 3-manifolds on 8 or 9 vertices are $J_2(8)$ and $J_2(9)$. Next we assume that $n \geq 10$, where $|W_2| = n - a \geq \lceil \frac{n}{2} \rceil - 2 > 2$. Hence the tree $\Delta[W_2\backslash\{v\}]$ has at least one edge, and thus there is a vertex $u_1 \in W_2$ such that $\deg_{\Delta[W_2]} u_1 = 1$. Let u_2 be the unique vertex in W_2 that is connected to u_1. Since $f_0(\mathrm{lk}_\Delta\, u_1) = a$, the vertex u_1 must be connected to all vertices in W_1 except for one vertex. We let z_1 be this vertex and denote the circle $\mathrm{lk}_{\mathrm{lk}_\Delta\, v}\, z_1$ by C_1. Since Δ is flag, $\mathrm{lk}_\Delta\, u_1 \supseteq \Delta[W_1\backslash\{z_1\}] = \mathrm{lk}_\Delta\, v - \{z_1\} * C_1$, and hence

$$\mathrm{lk}_\Delta\, u_1 = \big(\mathrm{lk}_\Delta\, v - \{z_1\} * C_1\big) \cup \big(\{u_2\} * C_1\big).$$

If $\{z_1\} \in \mathrm{lk}_\Delta\, u_2$, then $\mathrm{lk}_\Delta\, u_2 \supseteq C_1 * \{u_1, z_1\}$. Since $C_1 * \{u_1, z_1\}$ is a 2-sphere, it follows that $\mathrm{lk}_\Delta\, u_2 = C_1 * \{u_1, z_1\}$ and $f_0(C_1) = a - 2$. Hence $W_2 = \{u_1, u_2\}$ and $W_1 = V(C_1) \cup \{z_1\} \cup \{z_2\}$ for some vertex $z_2 \in W_1$, so that $\mathrm{lk}_\Delta\, v = \{z_1, z_2\} * C_1$. Now assume that $\{z_1\} \notin \mathrm{lk}_\Delta\, u_2$ and u_2 is connected to vertices $u_3, u_4, \cdots, u_k$ in $\Delta[W_2]$. Since C_1 is a circle in the 2-sphere $\mathrm{lk}_\Delta\, u_2$, the subcomplex $\mathrm{lk}_\Delta\, u_2\backslash V(C_1)$ has two contractible connected components. If there is a vertex u_i such that $\mathrm{lk}_{\mathrm{lk}_\Delta\, u_2}\, u_i = C_1$, then $\mathrm{lk}_\Delta\, u_2 \supseteq C_1 * \{u_1, u_i\}$ and hence this link is exactly $C_1 * \{u_1, u_i\}$. This implies that $\deg_{\Delta[W_2]} u_2 = 2$. Otherwise, if $\mathrm{lk}_{\mathrm{lk}_\Delta\, u_2}\, u_i \neq C_1$ for all $3 \leq i \leq k$, then each u_i is connected to at least one vertex in $\mathrm{lk}_\Delta\, v\backslash(V(C_1) \cup \{z_1\})$. Since $\mathrm{lk}_\Delta\, u_1 \supseteq \mathrm{lk}_\Delta\, v\backslash\{z_1\}$, it follows that the vertices u_1 and $u_3, \cdots, u_k$ are in the same connected component, and hence $\mathrm{lk}_\Delta\, u_2\backslash V(C_1)$ is connected, a contradiction.

By applying the above argument inductively, we obtain that $\Delta[W_2\backslash\{v\}]$ is a path graph $u_1, u_2, \cdots, u_{n-a-1}$, and there is a vertex z_2 in W_1 such that $\mathrm{lk}_\Delta\, u_1 = \{z_2, u_2\} * C_1$ and $\mathrm{lk}_\Delta\, v = C_1 * \{z_1, z_2\}$. Furthermore,

$C_1 \subseteq \mathrm{lk}_\Delta u_i$ for all $u_i \in W_2$. Then we let C_2 be the cycle graph $(v, z_2, u_1, u_2, \cdots, u_{n-a-1}, z_1)$. It follows that $\Delta = C_1 * C_2$. Since $a = |C_1| + 2 = \lfloor \frac{n+4}{2} \rfloor$ or $\lceil \frac{n+4}{4} \rceil$, C_1 and C_2 must be cycles of length $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$. This implies $\Delta = J_2(n)$.

By Lemma 2.2, the value of f_2 or f_3 determines f_1, and if either of them is maximal, then also f_1 is maximal. This yields the result. □

4 Counting edges of flag homology 5-manifolds

Recall that we use $J^*_m(n)$ to denote the suspension of $J_m(n-2)$. For even-dimensional flag homology spheres, the following is a special case of the last part of Conjecture 1.1:

Conjecture 4.1 *Fix $m \geq 1$. For every flag homology $2m$-sphere Δ on n vertices, we have $f_1(M) \leq f_1(J^*_m(n))$.*

Using the techniques similar to those in Section 3, we establish the following proposition.

Proposition 4.2 *Let Δ be a flag $(2m-1)$-manifold on n vertices. If Conjecture 4.1 holds for all flag homology $2i$-spheres with $1 \leq i \leq m-1$, then $f_1(\Delta) \leq f_1(J_m(n))$. Equality holds only when $\Delta = J_m(n)$.*

Proof: A careful adaptation of the argument in Lemma 3.1 and Theorem 1.3 yields the result. □

Proof of Theorem 1.4: The result follows from the fact that Conjecture 4.1 is known to hold in the case of dimension four (see Gal, 2005, Theorem 3.1.3). □

5 The face numbers of 3-dimensional flag Eulerian complexes

In Lemma 3.1, we established an upper bound on the number of edges for all flag normal 3-pseudomanifolds. In this section, we find tight upper bounds on the face numbers for all 3-dimensional flag Eulerian complexes. The proof relies on the following three lemmas.

Lemma 5.1 *Let Δ be a flag $(d-1)$-dimensional simplicial complex.*

(a) If σ_1 and σ_2 are two ridges that lie in the same facet σ in Δ, then the links of σ_1 and σ_2 are disjoint.

(b) If $\sigma = \tau_1 \sqcup \tau_2$ is a face in Δ, then $V(\mathrm{lk}_\Delta \tau_1) \cap V(\mathrm{lk}_\Delta \tau_2) = V(\mathrm{lk}_\Delta \sigma)$. In particular, if σ is a facet, then $f_0(\mathrm{lk}_\Delta \tau_1) + f_0(\mathrm{lk}_\Delta \tau_2) \leq f_0(\Delta)$.

The proof follows from the definition of flag complexes. We omit it for the sake of brevity. Lemma 5.1 part (b) implies that if Δ is a flag 3-dimensional simplicial complex and $\sigma \in \Delta$ is a facet, then $\sum_{e \subseteq \sigma} f_0(\mathrm{lk}_\Delta e) \leq 3f_0(\Delta)$, where the sum is over the edges of σ. The following lemma suggests a better estimate on $\sum_{e \subseteq \sigma} f_0(\mathrm{lk}_\Delta e)$ for an arbitrary flag weak 3-pseudomanifold Δ.

Lemma 5.2 *Let Δ be a flag weak 3-pseudomanifold on n vertices. Then for any facet $\sigma = \{v_1, v_2, v_3, v_4\}$ in Δ, $\sum_{e \subseteq \sigma} f_0(\mathrm{lk}_\Delta e) \leq n+16$, where the sum is over the edges of σ. If equality holds, then $\cup_{w \in \tau} V(\mathrm{lk}_\Delta w) = V(\Delta)$ for any ridge $\tau \subseteq \sigma$.*

Proof: Let $V_i = V(\mathrm{lk}_\Delta\, v_i)$ for $1 \leq i \leq 4$. By Lemma 5.1 part (b), for any distinct $1 \leq i, j \leq 4$, we have $V_i \cap V_j = V(\mathrm{lk}_\Delta\{v_i, v_j\})$ and $V_1 \cap V_2 \cap V_3 \cap V_4 = V(\mathrm{lk}_\Delta\, \sigma) = \emptyset$. Also since Δ is a weak 3-pseudomanifold, any ridge of Δ is contained in exactly two facets. Hence $V_i \cap V_j \cap V_k = V(\mathrm{lk}_\Delta\{v_i, v_j, v_k\})$ is a set of cardinality two. By the inclusiong-exclusion principle, we obtain that

$$\begin{aligned}\sum_{1\leq i<j\leq 4} |V_i \cap V_j| &= -|V_1 \cup V_2 \cup V_3 \cup V_4| + \sum_{1\leq i\leq 4} |V_i| + \sum_{1\leq i<j<k\leq 4} |V_i \cap V_j \cap V_k| - |V_1 \cap V_2 \cap V_3 \cap V_4| \\ &= \sum_{1\leq i\leq 4} |V_i| - |V_1 \cup V_2 \cup V_3 \cup V_4| + \binom{4}{3} \cdot 2 \\ &= (|V_1| + |V_2| - |V_1 \cup V_2|) + (|V_3| + |V_4| - |V_3 \cup V_4|) + |(V_1 \cup V_2) \cap (V_3 \cup V_4)| + 8 \\ &= |V_1 \cap V_2| + |V_3 \cap V_4| + |(V_1 \cup V_2) \cap (V_3 \cup V_4)| + 8.\end{aligned} \tag{2}$$

For simplicity, we denote the set $(V_1 \cup V_2) \cap (V_3 \cup V_4)$ as $\bar{V}$. Notice that by Lemma 5.1 part (b), any vertex $v \in \Delta$ belongs to at most one of the sets $V_1 \cap V_2$ and $V_3 \cap V_4$. We split the vertices of Δ into the following three types.

1. If $v \in V_1 \cap V_2$ and $v \notin V_3 \cup V_4$, or if $v \in V_3 \cap V_4$ and $v \notin V_1 \cup V_2$, then $v \notin \bar{V}$. Each of these vertices contributes 1 to the right-hand side of (2).

2. If $v \in V_i \cap V_j \cap V_k$ for some triple $\{i, j, k\} \subseteq [4]$, then v belongs to either $V_1 \cap V_2$ or $V_3 \cap V_4$, and $v \in \bar{V}$. By Lemma 5.1 part (a), every pair of ridges in σ has disjoint links. Since $|V_i \cap V_j \cap V_k| = 2$, the number of such vertices is exactly 8, and each of them contributes 2 to the right-hand side of (2).

3. If $v \notin V_1 \cap V_2$ and $v \notin V_3 \cap V_4$, then v contributes to the right-hand side of (2) at most 1. This case occurs only when $v \in \bar{V}$, that is, when v belongs to one of V_1 and V_2, and one of V_3 and V_4.

Hence $\sum_{\{i,j\}\subseteq[4]} |V_i \cap V_j| \leq n + 8 + 8 = n + 16$. Furthermore, if equality holds, then for every vertex v in Δ, either $v \in V_1 \cap V_2$, or $v \in V_3 \cap V_4$, or $v \in \bar{V}$. This implies that every vertex in Δ belongs to at least two of the four links $\mathrm{lk}_\Delta\, v_1, \cdots, \mathrm{lk}_\Delta\, v_4$. This proves the second claim. □

Lemma 5.3 *Let Δ be a flag weak 3-pseudomanifold on n vertices, and let $\sigma = \{v_1, v_2, v_3, v_4\}$ be an arbitrary facet of Δ. Then $\sum_{1\leq i\leq 4} f_0(\mathrm{lk}_\Delta\, v_i) \leq 2n+8$. If equality holds, then $\cup_{w\in\tau} V(\mathrm{lk}_\Delta\, w) = V(\Delta)$ for any ridge $\tau \subseteq \sigma$.*

Proof: The proof uses the inclusion-exclusion principle and Lemma 5.2. □

Now we are ready to prove the main result of this section.

Proof of Theorem 1.5: We denote the vertices of Δ by $v_1, v_2, \cdots, v_n$ and we let $a_i = f_0(\mathrm{lk}_\Delta\, v_i)$. Since $\mathrm{lk}_\Delta\, v_i$ is an Eulerian complex of dimension 2, the f-numbers of $\mathrm{lk}_\Delta\, v_i$ satisfy the relations

$$f_2 - f_1 + f_0 = 2, \;\; 3f_2 = 2f_1.$$

Hence $f_2(\mathrm{lk}_\Delta v_i) = 2a_i - 4$. By double counting, we obtain that

$$\sum_{\sigma\in\Delta,|\sigma|=4}\sum_{v\in\sigma} f_0(\mathrm{lk}_\Delta v) = \sum_{i=1}^{n} f_0(\mathrm{lk}_\Delta v_i)\cdot\#\{\sigma : v_i\in\sigma, |\sigma|=4\} = \sum_{i=1}^{n} a_i(2a_i-4). \tag{3}$$

By Lemma 5.3, the left-hand side of (3) is bounded above by $f_3(\Delta)(2n+8)$, which also equals $(f_1(\Delta)-n)(2n+8)$ by Lemma 2.2. However, since $2f_1(\Delta) = \sum_{i=1}^{n} f_0(\mathrm{lk}_\Delta v_1)$, the right-hand side of (3) is bounded below by $n\cdot\frac{2f_1(\Delta)}{n}\cdot\big(\frac{4f_1(\Delta)}{n}-4\big)$, and equality holds only if $a_i = \frac{2f_1(\Delta)}{n}$ for all $1\le i\le n$. Hence,

$$(f_1(\Delta)-n)(2n+8) \ge n\cdot\frac{2f_1(\Delta)}{n}\cdot\Big(\frac{4f_1(\Delta)}{n}-4\Big).$$

We simplify this inequality to get

$$(f_1(\Delta)-n)\Big(\frac{8}{n}f_1(\Delta)-(2n+8)\Big)\le 0.$$

Since $f_1(\Delta)\ge n$, it follows that $f_1(\Delta)\le\left\lfloor\frac{n^2}{4}\right\rfloor+n$, that is, $f_1(\Delta)\le f_1(J_2(n))$. Furthermore, if $f_1(\Delta)=\left\lfloor\frac{n^2}{4}\right\rfloor+n$, then there must be $\lceil\frac{n}{2}\rceil$ vertices such that $f_0(\mathrm{lk}_\Delta v)=\lfloor\frac{n}{2}\rfloor+2$, while the rest of vertices have $f_0(\mathrm{lk}_\Delta v)=\lceil\frac{n}{2}\rceil+2$. This proves our claim. □

The following corollary provides some further properties of the maximizers of the face numbers in the class of 3-dimensional flag Eulerian complexes. (We omit the proof.)

Corollary 5.4 *Let Δ be a 3-dimensional flag Eulerian complex on n vertices. If $f_1(\Delta)=f_1(J_2(n))$, then Δ and all of its vertex links are connected.*

The next lemma provides a sufficient condition for a flag complex to be the join of two of its links. The proof simply relies on Lemma 2.3 and properties of normal pseudomanifolds; we omit it here.

Lemma 5.5 *Let Δ be a flag $(d-1)$-dimensional simplicial complex. If $\sigma=\tau_1\cup\tau_2$ is a facet of Δ, where τ_1 is an i-face of Δ and τ_2 is a $(d-i-2)$-face of Δ, then $V(\mathrm{lk}_\Delta\tau_1)\cup V(\mathrm{lk}_\Delta\tau_2)=V(\Delta)$ implies that $\Delta\subseteq\mathrm{lk}_\Delta\tau_1 * \mathrm{lk}_\Delta\tau_2$. Moreover, if Δ is a flag normal $(d-1)$-pseudomanifold, then $V(\mathrm{lk}_\Delta\tau_1)\cup V(\mathrm{lk}_\Delta\tau_2)=V(\Delta)$ if and only if $\Delta=\mathrm{lk}_\Delta\tau_1 * \mathrm{lk}_\Delta\tau_2$.*

Remark 5.6 *The second result in Lemma 5.5 does not hold for flag weak pseudomanifolds, even assuming connectedness. Indeed, let $L_1,\cdots,L_4$ be four distinct circles of length ≥ 4. Then $\Delta=(L_1*L_3)\cup(L_2*L_3)\cup(L_1*L_4)$ is a flag weak 3-pseudomanifold. If τ_1 and τ_2 are edges in L_1 and L_3 respectively, then $\mathrm{lk}_\Delta\tau_1=L_3\sqcup L_4$ and $\mathrm{lk}_\Delta\tau_2=L_1\sqcup L_2$. Hence $V(\mathrm{lk}_\Delta\tau_1)\cup V(\mathrm{lk}_\Delta\tau_2)=V(\Delta)$. However, Δ is a proper subcomplex of $\mathrm{lk}_\Delta\tau_1 * \mathrm{lk}_\Delta\tau_2$.*

In Theorem 1.3, we proved that the maximizer of the face numbers is unique in the class of flag 3-manifolds on n vertices. Is this also true for 3-dimensional flag Eulerian complexes? Corollary 5.4 implies that if the case of equality is not a join of two circles, then some of its edge links are not connected. Motivated by the example in Remark 5.6, we construct a family of 3-dimensional flag Eulerian complexes on n vertices that have the same f-numbers as those of $J_2(n)$.

Example 5.7 *We write C_n to denote a circle of length n. Let $a_1, a_2, \cdots, a_s, b_1, b_2, \cdots, b_t \geq 4$ be integers such that*

$$\sum_{1\leq i\leq s} a_i = \left\lfloor \frac{n}{2} \right\rfloor, \text{ and } \sum_{1\leq j\leq t} b_j = \left\lceil \frac{n}{2} \right\rceil.$$

We claim that $\Delta = \cup_{1\leq i\leq s, 1\leq j\leq t}(C_{a_i} * C_{b_j})$ is flag and Eulerian, where all $C_\cdot$ are defined on disjoint vertex sets. Since the circles C_{a_i} and C_{b_j} are of length ≥ 4, it follows that Δ is flag. Also any ridge τ in Δ can be expressed as $\tau = \{v\} \cup e$, where $v \in C_{a_i}$ and the edge $e \in C_{b_j}$ (or $v \in C_{b_j}$ and $e \in C_{a_i}$) for some i, j. By the construction of Δ, the ridge τ is contained in exactly two facets $\{v, v'\} \cup e$ and $\{v, v''\} \cup e$ of Δ, where v' and v'' are neighbors of v in the circle C_{a_i} (or C_{b_j}). Hence the links of ridges in Δ are Eulerian. Since the edge links in Δ are either a circle or disjoint union of circles, and the vertex links in Δ are suspensions of disjoint union of circles, these links are also Eulerian. Finally, the vertices in C_{a_i} have degree $\lceil \frac{n}{2} \rceil + 2$ and the vertices in C_{b_j} have degree $\lfloor \frac{n}{2} \rfloor$, and thus $f_1(\Delta) = f_1(J_2(n))$. A simple computation also shows that $f_2(\Delta) = f_2(J_2(n))$ and $f_3(\Delta) = f_3(J_2(n))$. Hence $\chi(\Delta) = \chi(J_2(n)) = 0$ and Δ is Eulerian.

We denote the set of all complexes on n vertices constructed in Example 5.8 as $GJ(n)$. It turns out that $GJ(n)$ is exactly the set of maximizers of the face numbers in the class of flag 3-dimensional Eulerian complex on n vertices. To prove this, we begin with the following lemma.

Lemma 5.8 *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. If $f_1(\Delta) = f_1(J_2(n))$, then every vertex link is the suspension of disjoint union of circles.*

Proof: Applying the inclusion-exclusion principle on $V(\text{lk}_\Delta v_i)$, $1 \leq i \leq 3$, where v_1, v_2, v_3 are the vertices of a 2-face of Δ, and then using properties of Eulerian complexes yields the result. □

Theorem 5.9 *Let Δ be a flag 3-dimensional Eulerian complex on n vertices. If $f_1(\Delta) = f_1(J_2(n))$, then $\Delta \in GJ(n)$.*

Proof: By Lemma 5.8, we may assume that the link of vertex $v_1 \in \Delta$ is the join of C and two other vertices v_2, v_3, where C is the disjoint union of circles. Then again by Lemma 5.8, the link of vertex v_2 is also the suspension of C. If v'_1 is any vertex of C and its adjacent vertices in C are v'_2, v'_3, then by Lemma 2.3, $\Delta[V(C)] = C$, and it follows that $f_0(\text{lk}_\Delta v'_i \cap C) = 2$ for $i = 1, 2$. Hence for $1 \leq i, j \leq 2$,

$$f_0(\text{lk}_\Delta\{v_i, v'_j\}) = f_0(\text{lk}_\Delta v_i \cap \text{lk}_\Delta v'_j) \leq f_0(C \cap \text{lk}_\Delta v'_j) + 2 = 4.$$

Furthermore, $V(\text{lk}_\Delta\{v'_1, v'_2\})$ is disjoint from $V(\text{lk}_\Delta\{v_1, v_2\})$. So we obtain that

$$\sum_{e\subseteq\{v'_1, v'_2, v_1, v_2\}} f_0(\text{lk}_\Delta e) \leq n + 4 \cdot 4 = n + 16,$$

where the sum is over the edges of $\{v'_1, v'_2, v_1, v_2\}$. Since $f_1(\Delta) = f_1(J_2(n))$, by the proof of Theorem 1.5 and Lemma 5.2, it follows that this sum is exactly $n+16$. Hence $V(\text{lk}_\Delta\{v_1, v_2\}) \cup V(\text{lk}_\Delta\{v'_1, v'_2\}) = V(\Delta)$. By Lemma 5.5, $\Delta \subseteq \text{lk}_\Delta\{v_1, v_2\} * \text{lk}_\Delta\{v'_1, v'_2\}$. We count the number of edges in Δ to get

$$\begin{aligned} f_1(J_2(n)) = f_1(\Delta) &\leq f_1\big(\text{lk}_\Delta\{v_1, v_2\} * \text{lk}_\Delta\{v'_1, v'_2\}\big) \\ &= f_0\big(\text{lk}_\Delta\{v_1, v_2\}\big) \cdot f_0\big(\text{lk}_\Delta\{v'_1, v'_2\}\big) + n \\ &\leq f_1(J_2(n)). \end{aligned}$$

Thus $f_1(\Delta) = f_1(\mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\})$, and the edge links $\mathrm{lk}_\Delta\{v_1, v_2\}$ and $\mathrm{lk}_\Delta\{v_1', v_2'\}$ must be disjoint unions of circles on $\lceil \frac{n}{2} \rceil$ and $\lfloor \frac{n}{2} \rfloor$ vertices respectively. Since the flag complex Δ is determined by its graph, it follows that $\Delta = \mathrm{lk}_\Delta\{v_1, v_2\} * \mathrm{lk}_\Delta\{v_1', v_2'\}$, i.e., $\Delta \in GJ(n)$. $\square$

Remark 5.10 *Theorem 5.9 implies Theorem 1.3. This is because every 3-manifold is Eulerian and the only complex in $GJ(n)$ that is also a 3-manifold is $J_2(n)$.*

Remark 5.11 *The complexes from Example 5.8 form asymptotically the complete list of maximizers of the number of edges in the class of $K_{1,3,3}$-free graphs, see (Simonovits, 1966, Theorem 5). (Here $K_{r_1,r_2,\cdots,r_m}$ denotes the complete m-partite graph with r_i vertices of color i.) A more general result on extremal graphs not containing $K_{r_1,r_2,\cdots,r_m}$ can be found in Erdös and Simonovits (1972). Studying these extremal graphs is the main tool of Adamaszek and Hladký's work Adamaszek and Hladký (2015) on asymptotic upper bounds.*

6 Concluding Remarks

We close this paper with a few remarks and open problems.

As mentioned in the introduction, Klee (1964) verified that the Motzkin's UBC for polytopes holds for Eulerian complexes with sufficiently many vertices, and conjectured it holds for all Eulerian complexes. Can the flag upper bounds for spheres also be extended to Eulerian complexes? Motivated by Theorem 1.3 and Theorem 1.5, we posit the following conjecture in the same spirit as Problem 17(i) from Adamaszek and Hladký (2015):

Conjecture 6.1 *Let Δ be a flag $(2m-1)$-dimensional complex, where $m \geq 2$. Assume further that Δ is an Eulerian complex on n vertices. Then $f_i(\Delta) \leq f_i(J_m(n))$ for all $i = 1, \cdots, 2m-1$.*

Theorem 1.5 gives an affirmative answer in the case of $m = 2$ and $1 \leq i \leq 3$. The next case is $i = 1$ and $m = 3$. In this case, Theorem 1.4 verifies Conjecture 6.1 for flag 5-manifolds. At present other cases are completely open.

The above results and conjectures discuss odd-dimensional flag complexes. What happens in the even-dimensional cases? To this end, we pose the following strengthening of Conjecture 18 from Adamaszek and Hladký (2015).

Recall that $J_m^*(n) = \mathbb{S}^0 * C_1 * \cdots * C_m$, where $n \geq 4m+2$, each C_i is a circle of length either $\lceil \frac{n-2}{m} \rceil$ or $\lfloor \frac{n-2}{m} \rfloor$ so that the total number of vertices of $J_m^*(n)$ is n. Now we let $\mathcal{S}_n$ denote the set of flag 2-spheres on n vertices, and define

$$\mathcal{J}_m^*(n) := \{S * C_2 * \cdots * C_m \mid S \in \mathcal{S}_{V(C_1)+2}\}.$$

It is not hard to see that every element in $\mathcal{J}_m^*(n)$ is a flag $2m$-sphere.

Conjecture 6.2 *Let Δ be a flag homology $2m$-sphere on n vertices. Then $f_i(\Delta) \leq f_i(J_m^*(n))$ for all $i = 1, \cdots, 2m$. If equality holds for some $1 \leq i \leq 2m$, then $\Delta \in \mathcal{J}_m^*(n)$.*

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