Counting quadrant walks via Tutte's invariant method

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Abstract

In the 1970s, Tutte developed a clever algebraic approach, based on certain "invariants", to solve a func- tional equation that arises in the enumeration of properly colored triangulations. The enumeration of plane lattice walks confined to the first quadrant is governed by similar equations, and has led in the past decade to a rich col- lection of attractive results dealing with the nature (algebraic, D-finite or not) of the associated generating function, depending on the set of allowed steps.

We first adapt Tutte's approach to prove (or reprove) the algebraicity of all quadrant models known or conjectured to be algebraic (with one small exception). This includes Gessel's famous model, and the first proof ever found for one model with weighted steps. To be applicable, the method requires the existence of two rational functions called invariant and decoupling function respectively. When they exist, algebraicity comes out (almost) automatically.

Then, we move to an analytic viewpoint which has already proved very powerful, leading in particular to integral expressions of the generating function in the non-D-finite cases, as well as to proofs of non-D-finiteness. We develop in this context a weaker notion of invariant. Now all quadrant models have invariants, and for those that have in addition a decoupling function, we obtain integral-free expressions of the generating function, and a proof that this series is differentially algebraic (that is, satisfies a non-linear differential equation). This analytic approach solves as well the algebraic model left unsolved in the first part.

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